

EXISTENCE OF MINIMAL HYPERSURFACES IN COMPLETE MANIFOLDS OF FINITE VOLUME

GREGORY R. CHAMBERS AND YEVGENY LIOKUMOVICH

ABSTRACT. We prove that every complete non-compact manifold of finite volume contains a (possibly non-compact) minimal hypersurface of finite volume.

1. INTRODUCTION

By a result of Bangert and Thorbergsson (see [Th] and [Ba]) every complete surface of finite area contains a closed geodesic of finite length. In this article we generalize this result to higher dimensions.

Let M^{n+1} be a complete Riemannian manifold of dimension $n + 1$. For an open set $U \subset M$ define the relative width of U , denoted by $W_\partial(U)$, to be the supremum over all real numbers ω such that every Morse function $f : U \rightarrow [0, 1]$ has a fiber of volume at least ω .

Theorem 1.1. *Let M^{n+1} be a complete Riemannian manifold of dimension $n + 1$. Suppose M contains a bounded open set U with smooth boundary, such that $\text{Vol}_n(\partial U) \leq \frac{W_\partial(U)}{10}$. Then M contains a complete embedded minimal hypersurface Γ of finite volume. The hypersurface is smooth in the complement of a closed set of Hausdorff dimension $n - 7$.*

Remark 1.2. We make some remarks about Theorem 1.1:

1. The hypersurface Γ intersects a small neighbourhood of U . In fact, for any $\delta > 0$ there exists a finite volume minimal hypersurface that intersects the δ -neighbourhood of U (see Theorem 8.2 and Question 3 in Section 2.5).

2. If M is compact then Γ is compact. If M is not compact then Γ may or may not be compact. In Remark 8.3 we give an example, showing that one can not always expect to obtain a compact minimal hypersurface in a complete manifold of finite volume using a min-max argument.

3. We also obtain upper and lower bounds for the volume of Γ that depend on U (see Theorem 8.2).

The condition that there exists a subset U with $\mathcal{H}^n(\partial U) \leq \frac{W_\partial(U)}{10}$ is satisfied if tM has sublinear volume growth, that is, for some $x \in M$ we have $\liminf_{r \rightarrow \infty} \frac{\text{Vol}(B_r(x))}{r} = 0$. In particular, we have the following corollary.

Corollary 1.3. *Every complete non-compact Riemannian manifold M^{n+1} of finite volume contains a (possibly non-compact) embedded minimal hypersurface of finite volume. The hypersurface is smooth in the complement of a closed set of Hausdorff dimension $n - 7$.*

The proof is based on Almgren-Pitts min-max theory [Pi]. We use the version of the theory developed by De Lellis and Tasnady in [DT]. Instead of general sweepouts by integral flat cycles, the argument of [DT] allows one to consider sweepouts by hypersurfaces which are boundaries of open sets. This simplification is used in a crucial way in this paper. We consider a sequence of sweepouts of U and extract a sequence of hypersurfaces of almost maximal volume that converges to a minimal hypersurface. The main difficulty is to rule out the possibility that the sequence completely escapes into the “ends” of the manifold. Proposition 6.1 is the main tool which allows us to rule out this possibility. This Proposition allows us to replace an arbitrary family of hypersurfaces with a nested family of hypersurfaces which are level sets of a Morse function, increasing the maximal area by at most ε in the process. We use this Proposition together with some hands on geometric constructions to show that there exists a sequence of hypersurfaces that converges to a minimal hypersurface and the volume of their intersection with a small neighbourhood of U is bounded away from 0.

A number of results about existence of minimal hypersurfaces in non-compact manifolds have appeared recently. Existence results for minimal hypersurfaces (compact and non-compact) in certain classes of complete non-compact manifolds were proved by Gromov in [Gr]. This work was in part inspired by arguments in [Gr]. In [Gr] mean curvature of boundaries plays an important role. Our results do not depend on the curvature of the manifold or mean curvature of hypersurfaces in M . Existence of a compact embedded minimal surface in a hyperbolic 3-manifolds of finite volume was proved by Collin-Hauswirth-Mazet-Rosenberg in [CHMR]. In [Mo] Montezuma gave a detailed proof of the existence of embedded closed minimal hypersurfaces in non-compact manifolds containing a bounded open subset with mean-concave boundary, as well as satisfying certain conditions on the geometry at infinity. In particular, these manifolds have infinite volume. In [KZ] Ketover and Zhou proved a conjecture of Colding-Ilmanen-Minicozzi-White about the entropy of closed surfaces in \mathbb{R}^3 using a min-max argument for the Gaussian area functional on a non-compact space.

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2. STRUCTURE OF PROOF

We describe the idea of the proof.

2.1. Families of hypersurfaces and sweepouts. In this article we will be dealing with families of possibly singular hypersurfaces $\{\Gamma_t\}$. For the purposes of the introduction the reader may assume that each Γ_t is a boundary of a bounded open set Ω and has only isolated singularities of Morse type. In fact, Γ_t may differ from $\partial\Omega_t$ by a finite set of points. The precise definition of the hypersurfaces and the sense in which the family $\{\partial\Omega_t\}$ is continuous are described in Section 3. To follow the outline of the proof we only need to know that the areas of $\partial\Omega_{t_i}$ approach the area of $\partial\Omega_t$ and the volumes of $(\Omega_{t_i} \setminus \Omega_t) \cup (\Omega_t \setminus \Omega_{t_i})$ go to zero as $t_i \rightarrow t$. (We will use the word “volume” for the $(n+1)$ -dimensional Hausdorff measure and “area” for the n -dimensional Hausdorff measure.)

We will consider four types of special families of hypersurfaces, which we will call “sweepouts”. We will study the relationship between these four types of families and that will eventually lead us to the proof of Theorem 1.1. Slightly informally we describe them below.

1. An (ordinary) **sweepout** of a bounded set U is a family of hypersurfaces $\{\partial\Omega_t\}_{t \in [0,1]}$ with $\Omega_0 \cap U = \emptyset$ and $U \subset \Omega_1$.

2. A **good sweepout** of U is a sweepout $\{\Gamma_t\}$ with areas of Γ_0 and Γ_1 less than $5\mathcal{H}^n(\partial U)$.

The motivation for this definition is the following. In a “mountain pass” type argument we would like to apply a “pulling tight” deformation to a family $\{\Gamma_t\}$ so that hypersurfaces that have maximal area in the family converge (in a certain weak sense) to a stationary point of the area functional. When doing this we would like hypersurfaces at the “endpoints” Γ_0 and Γ_1 to stay fixed. We will consider sweepouts of sets with the property that every sweepout must contain a hypersurface of area much larger than the area of the boundary of U (see definition of a good set

below). The condition above guarantees that Γ_0 and Γ_1 do not have areas close to the maximum and so the pulling tight deformation will not affect them.

3. A **nested sweepout** of U is a sweepout $\{\partial\Omega_t\}_{t \in [0,1]}$ with $\Omega_s \subset \Omega_t$ for every $s \leq t$. Moreover, we have $\partial\Omega_t = f^{-1}(t)$ for some Morse function f . Nested sweepouts are a key technical tool in this paper.

4. A **relative sweepout** of U is a family of hypersurfaces $\{\Sigma_t\}$ with boundaries $\partial\Sigma_t \subset \partial U$ obtained from some nested sweepout $\{\Gamma_t\}$ of U by intersecting Γ_t with the closure of U , $\Sigma_t = \Gamma_t \cap cl(U)$.

2.2. Widths. For each notion of a sweepout we define a corresponding notion of **width**. If \mathcal{S} is a collection of families of hypersurfaces we set

$$W(\mathcal{S}) = \inf_{\{\Gamma_t\} \in \mathcal{S}} \sup_t \mathcal{H}^n(\Gamma_t)$$

Let $\mathcal{S}(U)$, $\mathcal{S}_\partial(U)$, $\mathcal{S}_g(U)$ and $\mathcal{S}_n(U)$ denote the collection of all sweepouts, relative sweepouts, good sweepouts and nested sweepouts correspondingly. We set $W(U) = W(\mathcal{S}(U))$ to be the width of U , $W_\partial(U) = W(\mathcal{S}_\partial(U))$ to be the relative width of U , $W_g(U) = W(\mathcal{S}_g(U))$ to be the good width of U and $W_n(U) = W(\mathcal{S}_n(U))$ to be the nested width of U .

Theorem 1.1 is a statement about a bounded open set $U \subset M$ with smooth boundary and the property that $\mathcal{H}^n(\partial U) \leq \frac{1}{10} W_\partial(U)$. A set satisfying this property will be called a **good set**. We will show that for a good set U we have the following relationships between the quantities $W(U)$, $W_\partial(U)$, $W_g(U)$ and $W_n(U)$:

$$(1) \quad W_\partial(U) \leq W_n(U) \leq W_\partial(U) + \mathcal{H}^n(\partial U)$$

$$(2) \quad W_n(U) = W(U)$$

$$(3) \quad W_g(U) = W(U)$$

The first inequality in (1) follows directly from the definition. The reason for the second inequality in (1) is also clear: to obtain a nested sweepout $\{\Gamma_t\}$ from a relative sweepout $\{\Sigma_t\}$ we can take a union of $\Sigma_t = \Gamma_t \cap cl(U)$ with a subset of the boundary ∂U (the subset varying based on Σ_t). Certain perturbation arguments will guarantee that a sufficiently regular nested sweepout can be obtained in this way. Note that this is also a good sweepout since it starts on a hypersurface of area 0 and ends on a hypersurface of area $\mathcal{H}^n(\partial U) < 5 \mathcal{H}^n(\partial U)$.

Equation (2) is proved in Proposition 6.1. In fact, (2) holds not only for good sets U , but for any bounded open set U with smooth boundary. The proof of (2) is the most technical part of this paper.

Equation (3) is proved below using methods from Section 7. The importance of these equations is the following: we will use (1) and (2) to prove (3); we will use (3) to prove Theorem 1.1.

2.3. Existence of a large slice intersecting U . Now we can outline the proof of Theorem 1.1. We would like to find a minimal hypersurface in M using a min-max argument, developed by Almgren [Al] and Pitts [Pi] and simplified by De Lellis - Tasnady [DT]. Let U be a good set. We choose a sequence of good sweepouts of U with the property that the area of the largest hypersurface converges to $W_g(U)$. We would like to extract an appropriate sequence of hypersurfaces whose areas converge to $W_g(U)$, and argue that they converge (as varifolds) to a minimal hypersurface.

The problem with this argument as it stands is that this sequence of hypersurfaces may drift off to infinity, and so strong convergence may not hold. To handle this issue, we will argue that this sequence of hypersurfaces can be chosen so that the intersection of every hypersurface with U is bounded away from 0. This “localization” statement will allow us to conclude that in the limit we obtain a minimal hypersurface with non-empty support in a small neighbourhood of U .

Proposition 2.1. *For every good set U there exists a positive constant $\varepsilon(U)$ which depends only on U such that the following holds. For every good sweepout $\{\Gamma_t\}$ of U with associated family of open sets $\{\Omega_t\}$, there is a surface $\Gamma_{t'}$ in the collection which has area at least $W_g(U)$, and such that $\mathcal{H}^n(\Gamma_{t'} \cap cl(U)) \geq \varepsilon(U)$.*

Theorem 1.1 will follow from by modifying arguments in [DT] (see Section 8). In the remainder of this section we focus on the proof of Proposition 2.1.

We explain how we choose $\varepsilon(U)$. In Section 7 (Lemma 7.1) we will show that for every U there exists $\varepsilon_0 > 0$ with the property that every Ω which intersects U in volume at most ε_0 or contains all of U except for a set of volume at most ε_0 can be deformed so that its boundary does not intersect U and the areas of the boundaries in the deformation process are controlled. Specifically, if $\mathcal{H}^{n+1}(\Omega \cap U) \leq \varepsilon_0$ then there exists a family $\{\Omega_t\}_{t \in [0,1]}$, such that $\Omega_0 \cap U = \emptyset$ and $\Omega_1 = \Omega$; if $\mathcal{H}^{n+1}(U \setminus \Omega) \leq \varepsilon_0$ then there exists a family $\{\Omega_t\}_{t \in [0,1]}$, such that $\Omega_1 \cap U = U$ and $\Omega_0 = \Omega$. In both cases the boundaries of Ω_t satisfy

$$(4) \quad \mathcal{H}^n(\partial \Omega_t) < \mathcal{H}^n(\partial \Omega) + 5 \mathcal{H}^n(\partial U)$$

Having fixed ε_0 with this property we define $\varepsilon(U) = \varepsilon(\varepsilon_0) > 0$ to be such that every Ω with $\min\{\mathcal{H}^{n+1}(\Omega \cap U), \mathcal{H}^{n+1}(U \setminus \Omega)\} \geq \varepsilon_0/2$ has $\mathcal{H}^n(\partial \Omega \cap U) > \varepsilon(U)$. Existence of such ε follows from the properties of the isoperimetric profile of U .

Suppose now that Proposition 2.1 fails for this value of $\varepsilon(U)$. Let $V(t) = \mathcal{H}^{n+1}(\Omega_t \cap U)$ and $A(t) = \mathcal{H}^n(\partial \Omega \cap U)$. V is a continuous function of t , $t \in [0, 1]$, but $A(t)$ may not

be continuous. However, the family $\{\partial\Omega_t\}$ can be perturbed to make $A(t)$ continuous. In the proof of Proposition 2.1 in section 7 we prove a weaker assertion that $A(t)$ is "roughly" continuous after a small perturbation, in the sense that the oscillation of A at a point t is at most $\varepsilon/10$; this turns out to be sufficient for what we need. For the purposes of this overview we will assume that $A(t)$ is actually continuous.

Continuity of A and V and the fact that $\{\partial\Omega_t\}$ is a sweepout imply that there exists an interval $[a, b] \subset [0, 1]$ with $\mathcal{H}^n(\partial\Omega_t \cap U) \geq \varepsilon$ for all $t \in [a, b]$; $\mathcal{H}^n(\partial\Omega_a \cap U) = \varepsilon$ and $\mathcal{H}^n(\partial\Omega_b \cap U) = \varepsilon$; $\mathcal{H}^{n+1}(\Omega_a \cap U) < \varepsilon_0/2$ and $\mathcal{H}^{n+1}(\Omega_b \cap U) > \mathcal{H}^{n+1}(U) - \varepsilon_0/2$. By our assumption this implies $\mathcal{H}^n(\partial\Omega_t) < W_g(U)$ for all $t \in [a, b]$. Since $\mathcal{H}^n(\partial\Omega_t)$ is a continuous function of t there exists a real number $\delta > 0$ such that $\partial\Omega_t$ has area at most $W_g(U) - \delta$ for $t \in [a, b]$.

Let $\tilde{U} = U \cap (\Omega_b \setminus cl(\Omega_a))$. The last paragraph implies that $W(\tilde{U}) \leq W_g(U) - \delta$. The boundary of \tilde{U} satisfies $\mathcal{H}^n(\partial\tilde{U}) \leq \mathcal{H}^n(\partial U) + 2\varepsilon$. Even though \tilde{U} may not be a good set we will show in Section 7 that (3) still holds for \tilde{U} . ($\mathcal{H}^n(\partial\tilde{U})$ may be larger than $1/10W_{\partial\tilde{U}}$, but it is still sufficiently small compared to $W_{\partial\tilde{U}}$ so that the proof of (3) goes through).

By (3) applied to \tilde{U} $W_g(\tilde{U}) = W(\tilde{U}) \leq W_g(U) - \delta$ and, hence, there exists a good sweepout $\{\partial\tilde{\Omega}_t\}_{t \in [0,1]}$ of \tilde{U} with areas of all hypersurfaces at most $W_g(U) - \delta/2$. By the definition of a sweepout $\tilde{\Omega}_0 \cap U \subset U \setminus \tilde{U} \subset (U \cap \Omega_a) \cup (U \setminus \Omega_b)$ and hence $\mathcal{H}^{n+1}(\tilde{\Omega}_0 \cap U) \leq \varepsilon_0$. Also, since $\{\partial\tilde{\Omega}_t\}$ is a good sweepout, $\partial\tilde{\Omega}_0$ has area at most $5\mathcal{H}^n(\partial U) - \delta/4$. By (4) we can deform $\tilde{\Omega}_0$ to a set that does not intersect U through open sets with boundary area at most $W_g(U) - \delta/4$. Similarly, we can deform $\tilde{\Omega}_1$ to an open set that contains U through open sets with boundary area at most $W_g(U) - \delta/4$. We conclude that there exists a sweepout of U by hypersurfaces of area at most $W_g(U) - \delta/4$. Hence, $W(U) \leq W_g(U) - \delta/4$, which contradicts (3). This finishes the proof of Proposition 2.1.

2.4. The good width equals width. In the rest of this section we describe how (3) follows from (1) and (2). The argument is illustrated in Figure 1. We start with a sweepout $\{\partial\Omega_t\}$ of a good set U by hypersurfaces of area at most $W(U) + \delta$. By (2) we can assume that $\{\partial\Omega_t\}$ is a nested sweepout. Next, we argue (cf. Lemma 7.4) that there is a hypersurface $\partial\Omega_{t'}$ with $t' \in [0, 1]$ such that $\mathcal{H}^n(\partial\Omega_{t'} \setminus U)$ has area comparable to that of the boundary of U . Indeed, by (1) there is a hypersurface with a large intersection with U , that is, $\mathcal{H}^n(\partial\Omega_{t'} \cap cl(U)) \geq W_n(U) - \mathcal{H}^n(\partial U)$. The complement then must satisfy $\mathcal{H}^n(\partial\Omega_{t'} \setminus cl(U)) \leq W(U) - W_n(U) + \mathcal{H}^n(\partial U) + \delta = \mathcal{H}^n(\partial U) + \delta$.

Now consider $\Omega_{t'} \setminus U$. Since $\{\partial\Omega_t\}$ is nested this set contains Ω_0 and is contained in Ω_1 . By the argument in the previous paragraph we have $\mathcal{H}^n(\partial(\Omega_{t'} \setminus U)) \leq 2\mathcal{H}^n(\partial U) + \delta$. Let A denote the infimal value of $\mathcal{H}^n(\partial\Omega)$ over all open sets Ω with $\Omega_0 \subset \Omega \subset$

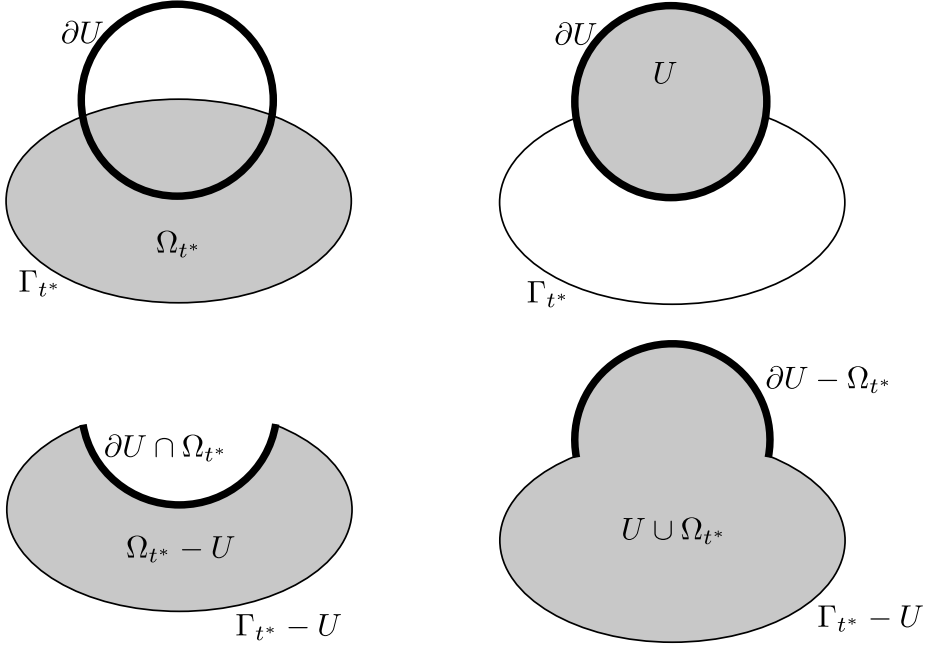


FIGURE 1. Cut and paste argument in the proof of 3

$\Omega_{t'} \setminus U$. Since $\Omega_{t'} \setminus U$ is one of such sets we have

$$A \leq 2\mathcal{H}^n(\partial U) + \delta$$

Let $\tilde{\Omega}$ denote a set as above with $\mathcal{H}^n(\partial\tilde{\Omega}) \leq A + \delta$. We replace sweepout $\{\partial\Omega_t\}$ with a new sweepout $\{\partial(\tilde{\Omega} \cup \Omega_t)\}$. Perturbation arguments will guarantee that we can smooth out the corners of these hypersurfaces to obtain a sufficiently regular family. This family starts on a surface $\partial\tilde{\Omega}$ of area less than $5\mathcal{H}^n(\partial U)$ and ends on Ω_1 . Moreover, it follows from the fact that $\partial\tilde{\Omega}$ is δ -nearly area minimizing hypersurface that the area of $\partial(\tilde{\Omega} \cup \Omega_t)$ is bounded by $W + 2\delta$ (cf. Lemma 5.1).

Similarly, we can replace this sweepout with a new sweepout that end on a hypersurface of area less than $5\mathcal{H}^n(\partial U)$, without increasing the areas of other hypersurfaces by more than δ . We conclude that $W_g(U) \leq W(U) + 3\delta$, but since $\delta > 0$ was arbitrary (3) follows.

The importance of nested sweepouts comes from the fact that it allows us to choose nearly minimizing hypersurfaces like $\partial\tilde{\Omega}$ and perform cut and paste procedures as above without increasing the area significantly. The ideas used in the proof of (2) and (3) go back to [CR] by the first author and Regina Rotman. In that article, the authors were interested in nested homotopies of curves, whereas here we use sufficiently regular cycles.

2.5. Open questions. We list some open questions related to Theorem 1.1.

1. For a positive real number α we say that U is an α -good set if $\mathcal{H}^n(\partial U) \leq \alpha W_\partial(U)$. Theorem 1.1 asserts that if a complete manifold M contains a $1/10$ -good set, then there is a minimal hypersurface of finite volume in M which intersects a small neighbourhood of U .

Question: What is the maximal value of α for which the conclusion of Theorem 1.1 holds? It is conceivable that it may be true for every positive $\alpha < 1$.

2. In [MN2] Marques and Neves show that a min-max minimal hypersurface has a connected component of Morse index 1, assuming that the manifold has no one-sided hypersurfaces (see [MR], [So], [Zh1], [Zh2] for previous results in that direction). Is it possible to adapt their arguments to construct a minimal hypersurface of finite volume and Morse index 1 for every complete manifold without one-sided hypersurfaces and satisfying the assumptions of Theorem 1.1?

3. In Theorem 8.2 we show that for an arbitrarily small $\delta > 0$ there exists a minimal hypersurface of finite volume intersecting the δ -neighbourhood of a good set U . Does there exist a minimal hypersurface of finite volume intersecting $cl(U)$? It is plausible that this result follows from a refinement of some of the arguments in Section 8 or from an appropriate compactness argument.

4. In [Gr] it is shown that if a non-compact manifold M does not admit a proper Morse function f , such that all non-singular fibers of f are mean-convex, then M contains a minimal hypersurface of finite volume. The following question was suggested to us by Misha Gromov:

Question: Do there exist manifolds of finite volume that admit a Morse function f , such that all non-singular level sets of f have positive mean curvature?

More generally, do there exist good sets U (in the sense defined in this paper) which admit Morse foliations by mean convex hypersurfaces (with boundaries of the hypersurfaces contained in the boundary of U)?

3. PRELIMINARIES

We begin with fixing notation and introducing several technical definitions which we will use throughout this article.

\mathcal{H}^k	k -dimensional Hausdorff measure
$cl(U)$	closure of the set U
$B_r(x)$	open ball of radius r centered at x
$N_r(U)$	the set $\{x \in M : d(x, U) < r\}$
$An(x, t_1, t_2)$	the open annulus $B_{t_2}(x) \setminus cl(B_{t_1}(x))$

Following De Lellis - Tasnady we make the following definitions.

3.1. Families of hypersurfaces and sweepouts.

Definition 3.1. Family of hypersurfaces A family $\{\Gamma_t\}$, $t \in [0, 1]$, of closed subsets of M with finite Hausdorff measure will be called a family of hypersurfaces if:

- (s1) For each t there is a finite set $P_t \subset M$ such that Γ_t is a smooth hypersurface in $M \setminus P_t$;
- (s2) $\mathcal{H}^n(\Gamma_t)$ depends smoothly on t and $t \rightarrow \Gamma_t$ is continuous in the Hausdorff sense;
- (s3) on any $U \subset\subset M \setminus P_{t_0}$, $\Gamma_t \rightarrow \Gamma_{t_0}$ smoothly in U as $t \rightarrow t_0$.

Definition 3.2. Sweepout Let U be an open subset of M . $\{\Gamma_t\}$, $t \in [0, 1]$, is a sweepout of M if it satisfies (s1)-(s3) and there exists a family $\{\Omega_t\}$, $t \in [0, 1]$, of open sets of finite Hausdorff measure, such that

- (sw1) $(\Gamma_t \setminus \partial\Omega_t) \subset P_t$ for any t ;
- (sw2) $\Omega_0 \cap U = \emptyset$ and $U \subset \Omega_1$;
- (sw3) $\mathcal{H}^{n+1}(\Omega_t \setminus \Omega_s) + \mathcal{H}^{n+1}(\Omega_s \setminus \Omega_t) \rightarrow 0$ as $t \rightarrow s$.

For a sweepout $\{\Gamma_t\}$ we will say that $\{\Omega_t\}$ is the corresponding family of open sets if it satisfies (sw1) - (sw3).

Definition 3.3. Good sweepouts, nested sweepouts and relative sweepouts

A **good sweepout** $\{\Gamma_t\}$ is a sweepout of U which in addition satisfies:

$$(\text{sw}_g) \quad \mathcal{H}^n(\Gamma_0) \leq 5 \mathcal{H}^n(\partial U) \text{ and } \mathcal{H}^n(\Gamma_1) < 5 \mathcal{H}^n(\partial U).$$

A **nested sweepout** $\{\Gamma_t\}$ is a sweepout of U which in addition satisfies:

(sw_n) there exists a Morse function $f : M \rightarrow [-1, \infty)$, such that $\Gamma_t = f^{-1}(t)$, $t \in [0, 1]$; the corresponding family of open sets is given by $\Omega_t = f^{-1}((-\infty, t))$.

Suppose ∂U is a smooth manifold and $\{\Gamma_t\}$ is a nested sweepout of U with the corresponding family of open sets $\{\Omega_t\}$. Set $\Sigma_t = (cl(U) \cap \Gamma_t)$. We will say that $\{\Sigma_t\}$ is a **relative sweepout** of U .

Definition 3.4. Widths and good sets As described in Section 2 the widths $W(U)$, $W_\partial(U)$, $W_g(U)$ and $W_n(U)$ are defined as the min-max quantities corresponding to sweepouts, relative sweepouts, good sweepouts and nested sweepouts respectively.

A good set $U \subset M$ is a bounded open set with smooth boundary and $\mathcal{H}^n(\partial U) \leq \frac{1}{10} W_\partial(U)$.

3.2. Smoothing corners. Let $N \subset M$ be an open subset and suppose $\Sigma_1 \subset \partial N$ and $\Sigma_2 \subset \partial N$ are n -dimensional submanifolds of M , such that the interiors of Σ_1 and

Σ_2 are disjoint, $\Sigma_1 \cup \Sigma_2 = \partial N$ and $\partial \Sigma_1 \cap \partial \Sigma_2 = C$ is a compact $(n-2)$ -dimensional submanifold of M .

We say that ∂N is a manifold with corner C if for every sufficiently small neighbourhood U of a point $x \in C$ there exists a diffeomorphism ϕ from U to \mathbb{R}^{n+1} with $\phi(N) = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $\phi(\Sigma_1) = \{x_1 = 0\}$, $\phi(\Sigma_2) = \{x_2 = 0\}$ and $C = \{x_1 = x_2 = 0\}$.

There is a standard construction of smoothing (or straightening) the corner C of a manifold with corner (see [Mu, Section 7.5]). We briefly describe it here, because we use it several times in this paper.

Fix $\delta > 0$. We construct a smooth hypersurface $\Sigma \subset cl(N)$, such that Σ coincides with ∂N outside of $N_\delta(C)$. For each $x \in C$ let $\theta(x) \in (0, 2\pi)$ denote the angle between hyperplanes $T_x \Sigma_1$ and $T_x \Sigma_2$ inside tangent space $T_x M$. Define cylindrical coordinates $y = (x, \theta, r)$ on $cl(N_\delta(C) \cap N)$, where $x \in C$, r denotes the radial distance to C and $\theta \in [0, \theta(x)]$ denotes the angle that a minimizing geodesic from C to y makes with the hyperplane $T_x \Sigma_1$.

Let $\gamma_x(t)$ be a family of smooth convex functions defined on $[0, \theta(x)]$ with $\min_t \gamma_x(t) = \delta/4$, $\max_t \gamma_x(t) = \gamma_x(0) = \gamma_x(\theta(x)) = \delta/2$ and $\frac{d}{dk} \gamma_x(t) = \infty$ for all $k > 0$ as t approaches 0 or $\theta(x)$. We define Σ in $N_\delta(C)$ by setting $\Sigma \cap N_{\delta/2}(C) = \{(x, \theta, r) : r = \gamma_x(\theta)\}$ for $\theta \in [0, \theta(x)]$ and $r \leq \delta/2$ and $\Sigma \cap (N_\delta(C) \setminus N_{\delta/2}(C)) = (\Sigma_1 \cup \Sigma_2) \cap (N_\delta(C) \setminus N_{\delta/2}(C))$.

We make several observations about this construction.

1. Different smoothings Σ corresponding to different choices of the convex functions $\gamma_x(t)$ are all isotopic.
2. For any $\varepsilon > 0$ functions $\gamma_x(t)$ can be chosen in such a way that $\mathcal{H}^n(\Sigma) < \mathcal{H}^n(\partial N) + \varepsilon$.
3. Smoothing can be done parametrically. Given a foliation of a subset of M by hypersurfaces with corners the above construction can be applied to the whole family in such a way that we obtain a foliation by a family of smooth hypersurfaces.
4. For all $\delta > 0$ sufficiently small there exists a choice of Σ and a constant c that depends on M , N and C , so that $\mathcal{H}^n(N_{10\delta}(C) \cap \partial N_{2\delta}(\Sigma)) \leq c\delta$.

The last observation will be important in the proof of Lemma 4.3.

It will be convenient to introduce one more definition.

Definition 3.5. Let $\Omega \subset M$ be a bounded open subset and $\partial \Omega$ is a manifold with corner and $\delta > 0$. We will say that $\Omega_{+\delta}$ is an outward δ -perturbation of Ω if the following holds:

- (1) $\Omega \subsetneq \Omega_{+\delta} \subset N_\delta(\Omega)$;
- (2) there exists a nested family of open sets $\{\Xi_t\}_{t \in [0,1]}$ and a smooth isotopy $\Sigma_t = \partial \Xi_t$, such that Σ_0 is a smoothing of $\partial \Omega$, $\Xi_1 = \Omega_{+\delta}$ and $\mathcal{H}^n(\Sigma_t) < \mathcal{H}^n(\partial \Omega) + \delta$ for all $t \in [0, 1]$.

We will say that $\Omega_{-\delta}$ is an inward δ -perturbation of Ω if the following holds:

- (1)' $\Omega \setminus N_\delta(\partial\Omega) \subset \Omega_{-\delta} \subsetneq \Omega$;
- (2)' there exists a nested family of open sets $\{\Xi_t\}_{t \in [0,1]}$ and a smooth isotopy $\Sigma_t = \partial\Xi_t$, such that Σ_1 is a smoothing of $\partial\Omega$, $\Xi_0 = \Omega_{-\delta}$ and $\mathcal{H}^n(\Sigma_t) < \mathcal{H}^n(\partial\Omega) + \delta$ for all $t \in [0, 1]$.

4. MORSE FOLIATIONS WITH CONTROLLED AREA OF FIBERS.

Here we present several results about concatenating different Morse foliations and controlling areas of fibers of Morse functions.

For PL Morse functions Sabourau proved similar results in [Sa].

4.1. Gluing Morse foliations. Let $N \subset M$ be a compact submanifold of M with boundary. We will say that a Morse function $f : N \rightarrow \mathbb{R}$ is ∂ -transverse if

- (1) there exists an extension \tilde{f} of f to an open neighbourhood of N in M , such that all critical points are isolated, non-degenerate and lie in the interior of N ;
- (2) the restriction of f to ∂N is a Morse function.

Lemma 4.1. *Let $N \subset M$ be a compact submanifold with non-empty boundary and $f : N \rightarrow [a, b]$ be a ∂ -transverse Morse function. Let Σ be a closed submanifold of ∂N .*

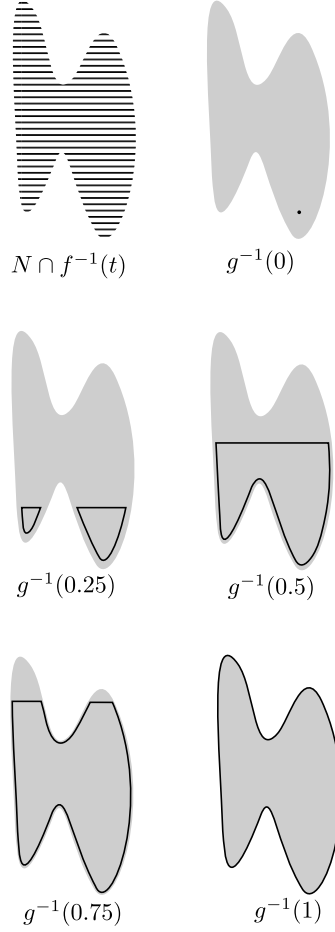
For every $\varepsilon > 0$ there exists a Morse function $g : N \rightarrow [a, b]$, such that the following holds:

- (1) $g^{-1}(b) = \Sigma$;
- (2) $f^{-1}([a, t]) \subset N_{\varepsilon/2}(g^{-1}([a, t])) \subset N_\varepsilon(f^{-1}([a, t]))$;
- (3) $\mathcal{H}^n(g^{-1}(t)) \leq \mathcal{H}^n(\partial f^{-1}([a, t])) + \varepsilon$;
- (4) If $\text{dist}(x, f(\Sigma)) > \varepsilon$ then $f^{-1}(x) = g^{-1}(x)$.

Proof. The idea of the proof is shown in Figure 2 We will define a singular foliation Σ_t , $t \in [0, 1]$, of N with only finitely many singular leaves that have non-degenerate singularities and with $\Sigma_1 = \Sigma$. It follows then that there exists a Morse function $g(x)$ with $g^{-1}(t) = \Sigma_t$. We will prove that this foliation satisfies the desired upper bound on the area. The surfaces in the foliation will coincide with $f^{-1}(t)$ whenever $f^{-1}(t)$ is sufficiently far from Σ and so (4) will also follow.

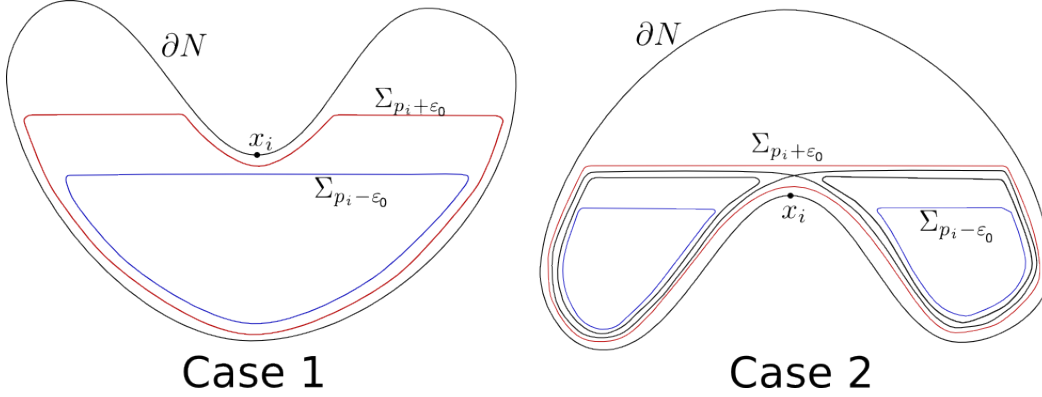
Choose $r_0 \in (0, \varepsilon)$, be sufficiently small, so that the tubular neighbourhood $U = N_{2r_0}(\Sigma) \cap N$ does not intersect critical points of f and there exists a diffeomorphism ϕ from $\Sigma \times [0, 2r_0)$ to U . Let $\phi(x, r)$, $x \in \Sigma$, $r \in [0, r_0)$ denote the normal coordinates on U . For r_0 sufficiently small we may assume that $\mathcal{H}^n((\Sigma, r)) \leq \mathcal{H}^n(\Sigma) + \frac{\varepsilon}{2}$ for $r \in [0, r_0]$. Let $U_r = \{\phi(x, r') : r' \leq r\}$. Let $\varepsilon_0 = \varepsilon_0(r_0) > 0$ be a small constant to be specified later and satisfying $\varepsilon_0 \rightarrow 0$ for $r_0 \rightarrow 0$.

Let $p_0 < \dots < p_k$ be critical values of $f|_\Sigma$. First we define a singular foliation Σ_t , $t \notin \cup_i(p_i - \varepsilon_0, p_i + \varepsilon_0)$. Let $\bar{\Sigma}_t = \partial(f^{-1}([0, t]) \setminus U_{(1-t)r_0})$. If t is a singular value of f

FIGURE 2. Constructing a singular foliation of N .

then $\bar{\Sigma}_t$ has a Morse type singularity at the singular point s of f in the interior of N . Since t is at least ε_0 away from singular values of $f|_{\Sigma}$ we have that $f^{-1}(t)$ intersects $\phi(\Sigma, (1-t)r_0)$ transversally. Hence, $\bar{\Sigma}_t \setminus s$ is a manifold with corners. There exists a smoothing of the corners, so that the new foliation $\{\Sigma_t\}$ coincides with $\{\bar{\Sigma}_t\}$ outside of a small neighbourhood of $V_t = f^{-1}(t) \cap \phi(\Sigma, (1-t)r_0)$ and is smooth in V_t . As discussed in subsection 3.2 we can choose it so that $\mathcal{H}^n(\Sigma_t) - \mathcal{H}^n(\bar{\Sigma}_t)$ is arbitrarily small.

Now we construct the foliation for $t \in (p_i - \varepsilon_0, p_i + \varepsilon_0)$. Let $x_i \in \Sigma$ be the critical point of $f|_{\Sigma}$ with $f(x_i) = p_i$. Outside of a small neighbourhood of x_i we can define Σ_t in the same way as above, since $f^{-1}(t)$ intersects $\phi(\Sigma, (1-t)r_0)$ transversally and a smoothing of the corners is well-defined.

FIGURE 3. Procedure for dealing with singularities on the boundary of N .

In the neighbourhood of a critical point x_i we define the foliation by considering two cases (see Figure 3). Let n_i denote the inward pointing unit normal at x_i and set $s_i = \langle \frac{\nabla f}{|\nabla f|}(p_i), n_i \rangle$. The two cases will depend on the sign of s_i .

Let $y_i = \phi(x_i, (1 - p_i)r_0)$. There exists a choice of coordinates $u = (u_1, \dots, u_{n+1})$ in the neighbourhood of y_i so that in these coordinates we have $f(u) = u_{n+1} + f(y_i)$. Let λ denote the index of x_i . Let $P_\lambda(u_1, \dots, u_n) = -u_1^2 - \dots - u_\lambda^2 + u_{\lambda+1}^2 + \dots + u_n^2$. Up to a bilipschitz diffeomorphism of the neighbourhood of y_i , the foliation $\{\phi(\Sigma, (1 - t')r_0)\}$, $t' \in (p_i - \varepsilon_0, p_i + \varepsilon_0)$, will coincide with the foliation $\{u_{n+1} = P_\lambda(u_1, \dots, u_n) - s_i t\}$, $t \in (-\varepsilon_0, \varepsilon_0)$.

Case 1: $s_i = -1$. There exists a smoothing of the corners for Σ_t so that as t approaches p_i from above and below surface Σ_t is a graph over $\{u_{n+1} = 0\}$ hyperplane in the neighbourhood of y_i . There exists a small $\delta > 0$ and a foliation $\{\Gamma_t\}$ of the neighbourhood of y_i so that $\Gamma_t = \{u_{n+1} = P_\lambda(u_1, \dots, u_n) + t\}$ for $u_1^2 + \dots + u_n^2 < \delta/3$ and Γ_t is a graph of $u_{n+1} = t$ for $u_1^2 + \dots + u_n^2 > 2\delta/3$. The foliation $\{\Gamma_t\}$ extends the foliation $\{\Sigma_t\}$ to the neighbourhood of the critical point x_i .

Case 2: $s_i = 1$. Let $\Pi_t = \{u_{n+1} = t\} \cap \{P_\lambda(u_1, \dots, u_n) \leq 2t\}$ and $Q_t = \{u_{n+1} = P_\lambda(u_1, \dots, u_n) - t\} \cap \{u_{n+1} \leq t\}$. After a bilipschitz diffeomorphism in the neighbourhood of y_i we may assume that the foliation $\{\Sigma_{t'}\}$ is given by the smoothing of the union $\Pi_t \cup Q_t$. By standard Morse theory arguments (see Section 3 of [Mi1] and Section 3 of [Mi2]) $\Pi_\delta \cup Q_\delta$ is obtained from $\Pi_{-\delta} \cup Q_{-\delta}$ by surgery of type $(\lambda, n+1-\lambda)$ and there exists an elementary cobordism between them of index λ . This cobordism gives the desired foliation in the neighbourhood of the critical point.

Observe that in the above operations we applied bilipschitz diffeomorphisms on some small neighbourhood, possibly increasing the areas of hypersurfaces by some

controlled constant factor (independent of the size of the neighbourhood). By choosing the neighbourhood to be sufficiently small we ensure that the areas do not increase by more than ε . □

We will also need a slightly different version of this lemma for a non-compact submanifold N .

Lemma 4.2. *Let $N \subset M$ be a not necessarily compact submanifold with non-empty boundary and $f : N \rightarrow (-\infty, b]$ be a proper Morse function, which is ∂ -transverse. Let Σ be a compact submanifold of ∂N .*

For every $\varepsilon > 0$ there exists a Morse function $g : N \rightarrow (-\infty, b]$, such that the following holds:

- (1) $g^{-1}(b) = \Sigma$;
- (2) $f^{-1}((-\infty, t)) \subset N_{\varepsilon/2}(g^{-1}((-\infty, t))) \subset N_{\varepsilon}(f^{-1}((-\infty, t)))$;
- (3) $\mathcal{H}^n(g^{-1}(t)) \leq \mathcal{H}^n(\partial f^{-1}((-\infty, t))) + \varepsilon$;
- (4) *If $\text{dist}(x, f(\Sigma)) > \varepsilon$ then $f^{-1}(x) = g^{-1}(x)$.*

Proof. Let a be such that $f(N_{\varepsilon}\Sigma) \subset [a + \varepsilon, b]$. Since function f is proper we have that $N' = f^{-1}([a, b])$ is compact. We apply Lemma 4.1 to N' to obtain function g . We set $g(x) = f(x)$ for x not in N' and the lemma follows. □

4.2. Gluing Morse foliations on a manifold separated by a hypersurface transverse to the boundary. We will also need the following lemma for gluing two Morse foliations on a manifold with boundary separated by a hypersurface which is transversal to the boundary.

Lemma 4.3. *Let N be a manifold with compact boundary ∂N and Σ be a hypersurface with $\partial\Sigma \subset \partial N$ and such that Σ intersects ∂N transversally. Suppose $N \setminus \Sigma = V_1 \sqcup V_2$. For every $\varepsilon > 0$ there exist open sets with smooth boundary Ω_1 and Ω_2 and a Morse function $f : \text{cl}(N \setminus (\Omega_1 \cup \Omega_2)) \rightarrow [0, 1]$, such that the following holds:*

- (1) Ω_1 is an inward ε -perturbation of V_1 ; Ω_2 is an inward ε -perturbation of V_2 .
- (2) $f^{-1}(0) = \partial\Omega_1 \cup \partial\Omega_2$ and $f^{-1}(1) = \partial N$;
- (3) $\mathcal{H}^n(f^{-1}(t)) \leq \mathcal{H}^n(\partial N) + 2\mathcal{H}^n(\Sigma) + \varepsilon$.

Proof. The idea of the proof of this lemma is shown in Figure 4. Fix $\delta > 0$ to be specified later.

Note that ∂V_i is a manifold with a corner $\Sigma \cap \partial V_i$. Let $V'_i \subset V_i$ be a submanifold with $\partial V'_i$ a smoothing of ∂V_i . We have that V_i and V'_i coincide outside of $N_{\delta/2}(\partial V_i \cap \partial\Sigma)$. Let $\Omega_i = V'_i \setminus N_{\delta/2}(\partial V'_i)$.

Let $d : M \setminus (\Omega_1 \cup \Omega_2) \rightarrow [0, \infty)$ denote the distance from x to $\Omega_1 \cup \Omega_2$. Function d is 1-Lipschitz, but it may not be smooth. However, it is well-known ([GW]) that for every $\varepsilon > 0$ function d may be approximated by a Morse function f with $1 - \varepsilon <$

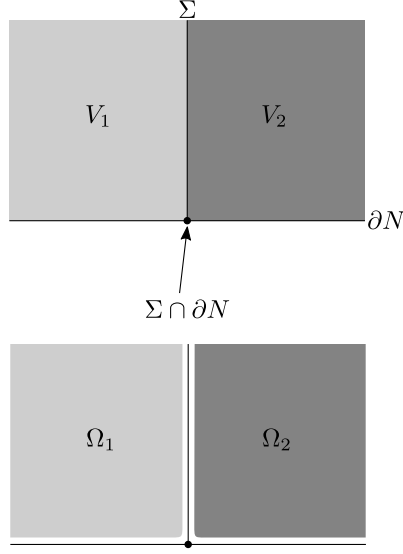


FIGURE 4. Gluing two submanifolds using a Morse foliation.

$|\nabla f| < 1 + \varepsilon$. We choose such an approximation and consider level sets $f^{-1}(t)$, $t \in [0, 2\delta]$. Define $\Omega_3 = \Omega_1 \cup \Omega_2 \cup f^{-1}([0, 2\delta])$.

By curvature comparison arguments from [HK] applied to function f we know that $\frac{d}{dt} \mathcal{H}^n(f^{-1}(t))$ only depends on the Ricci curvature of $N_{2\delta}(\partial N)$ and the mean curvature of $\partial(\Omega_1 \cup \Omega_2)$. The mean curvature of $\partial(\Omega_1 \cup \Omega_2)$ in turn depends on the mean curvatures of ∂N , Σ and the choice of smoothing of the corners for V_1 and V_2 . As observed in subsection 3.2 we may assume that the contribution that comes from the smoothing of the corners is negligible for sufficiently small δ .

It follows that we can find a $\delta > 0$ so that $\mathcal{H}^n(f^{-1}(t)) \leq \mathcal{H}^n(\partial V_1) + \mathcal{H}^n(\partial V_2) + \varepsilon$.

The above construction does not yet give us what we want because $f^{-1}(1) = \partial \Omega_3$, which sits slightly outside of ∂N . To fix this we construct function f as above not for N , but for $N' = N \setminus N_{3\delta}(\partial N)$, for some suitable sufficiently small choice of δ to ensure that $\partial N'$ is smooth and intersects Σ transversally. Then Ω_3 sits inside N and there exists a nested isotopy from $\partial \Omega_3$ to ∂N . \square

5. SPLITTING AND EXTENSION LEMMAS

In this section we prove two important lemmas for nested sweepouts which we will use in sections “Nested sweepouts” and “No escape to infinite”.

Lemma 5.1. *Suppose that $f : M \rightarrow [-1, \infty)$ is a Morse function and $\{\Gamma_t\} = \{f^{-1}(t)\}_{t \in [0, 1]}$ is a nested family of hypersurfaces of area $\leq A$ with associated open sets $\{\Omega_t\} = \{f^{-1}((-\infty, t))\}$.*

I. Additionally, suppose that Ω is a bounded open set with boundary Γ a smooth embedded manifold such that

- (1) $\Omega \subset \Omega_1$;
- (2) *There is an $\varepsilon > 0$ such that for every Ω' with $\Omega \subset \Omega' \subset \Omega_1$ we have $\mathcal{H}^n(\Gamma) < \mathcal{H}^n(\partial\Omega') + \varepsilon/4$.*

Then we can find a nested family $\tilde{\Gamma}_t$ and an associated family of open sets $\tilde{\Omega}_t$ such that $\tilde{\Omega}_0 \subset \Omega_0$, $\tilde{\Gamma}_1 = \Gamma$, and every hypersurface has area at most $A + \varepsilon$. Furthermore, if $\Omega_0 \subset \Omega$, then $\tilde{\Gamma}_0 = \Gamma_0$.

II. Suppose that, instead of properties (1) and (2) above, the following are true:

- (1)' $\Omega_0 \subset \Omega$;
- (2)' *There is an $\varepsilon > 0$ such that for every Ω' with $\Omega_0 \subset \Omega' \subset \Omega$ we have $\mathcal{H}^n(\Gamma) < \mathcal{H}^n(\partial\Omega') + \varepsilon/4$.*

Then we can find a nested family $\tilde{\Gamma}_t$ and an associated family of open sets $\tilde{\Omega}_t$ such that $\Omega_1 \subset \tilde{\Omega}_1$, $\tilde{\Gamma}_0 = \Gamma$, and every hypersurface has area at most $A + \varepsilon$. Furthermore, if $\Omega \subset \Omega_1$, then $\tilde{\Gamma}_1 = \Gamma_1$.

Proof. The argument is demonstrated in Figure 5. We begin with a proof of the first half of this lemma.

We consider two cases. Suppose first that $\Omega \subset \Omega_0$. For a sufficiently small $\delta > 0$ the function $g : cl(N_\delta(\Gamma) \cap \Omega) \rightarrow [0, 1]$ given by $g(x) = \frac{1}{\delta} dist(x, \Gamma)$ is a smooth function with no critical points and $\tilde{\Gamma}_t = g^{-1}(t)$ a hypersurface of area at most $\mathcal{H}^n(\Gamma) + \varepsilon/2$. By condition (2) $\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\partial\Omega_0) + \varepsilon/4$ and so $\mathcal{H}^n(\tilde{\Gamma}_t) \leq A + \varepsilon$. We extend g to a Morse function on M in an arbitrary way. $\{\tilde{\Gamma}_t\}$ is a nested family satisfying the conclusions of the theorem.

Suppose now that $\Omega \setminus \Omega_0 \neq \emptyset$. Make a small perturbation to the hypersurface $\Gamma = \partial\Omega$, so that $f|_{cl(\Omega)}$ is Morse and (1) and (2) are still satisfied, possibly replacing $\varepsilon/4$ in (2) by $\varepsilon/2$.

Consider f restricted to Ω . After composing with a diffeomorphism of $[-1, \infty)$ we may assume that $f(cl(\Omega)) \subset [-1, 1]$ and $f(cl(\Omega \setminus \Omega_0)) = [0, 1]$. We apply Lemma 4.1 with $N = \Omega$ and $\Sigma = \Gamma$ to obtain a Morse function $g : \Omega \rightarrow [-\frac{1}{2}, 1]$, such that $g^{-1}(-1)$ is a point in Ω , $g^{-1}(1) = \Gamma$ and $\mathcal{H}^n(g^{-1}(t)) \leq \mathcal{H}^n(\partial(f^{-1}([-1, t]) \cap \Omega) + \varepsilon/2$. It follows that $\mathcal{H}^n(g^{-1}(t)) \leq \mathcal{H}^n(f^{-1}(t) \cap \Omega) + \mathcal{H}^n(f^{-1}([-1, t]) \cap \Gamma) + \varepsilon/2$. Furthermore, we have $g^{-1}([-1, 0)) \subset N_\varepsilon(\Omega \cap \Omega_0)$. After a small perturbation of the function g we may assume that $g^{-1}([-1, 0)) \subset (\Omega \cap \Omega_0)$. We extend g to a Morse function on M in an arbitrary way. We claim that $\Gamma_t = g^{-1}(t)$ for $t \in [0, 1]$ is the desired nested family. The only thing left to prove is an upper bound for the areas of $\tilde{\Gamma}_t$.

For any smooth hypersurface Σ_t obtained by a small perturbation of $\partial(\Omega \cup \Omega_t)$ we have $\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\Sigma_t) + \varepsilon/4$ by (2). It follows that

$$\mathcal{H}^n(\Gamma) \leq \mathcal{H}^n(\partial(\Omega \cup \Omega_t)) + \varepsilon/2$$

Since $\partial(\Omega \cup \Omega_t) = (\Gamma_t \cap \Omega) \cup (\Gamma \setminus \Omega_t)$ we have

$$\begin{aligned} \mathcal{H}^n(\Gamma \cap \Omega_t) + \mathcal{H}^n(\Gamma \setminus \Omega_t) &\leq \mathcal{H}^n(\Gamma_t \setminus \Omega) + \mathcal{H}^n(\Gamma \setminus \Omega_t) + \varepsilon/2 \\ \mathcal{H}^n(\Gamma \cap \Omega_t) &\leq \mathcal{H}^n(\Gamma_t \setminus \Omega) + \varepsilon/2 \end{aligned}$$

By Lemma 4.1 we have

$$\begin{aligned} \mathcal{H}^n(\tilde{\Gamma}) &\leq \mathcal{H}^n(\Gamma_t \cap \Omega) + \mathcal{H}^n(\Gamma \cap \Omega_t) + \varepsilon/2 \\ &\leq \mathcal{H}^n(\Gamma_t \cap \Omega) + \mathcal{H}^n(\Gamma_t \setminus \Omega) + \varepsilon \\ &\leq \mathcal{H}^n(\Gamma_t) + \varepsilon \leq A + \varepsilon \end{aligned}$$

If $\Omega_0 \subset \Omega$, then by choosing sufficiently small $\varepsilon > 0$ and applying Lemma 4.1 (4) we have $\tilde{\Gamma}_0 = f^{-1}(0) = \Gamma_0$.

The proof of the second half is similar.

If $\Omega_1 \subset \Omega$ we define the desired nested family $\{\tilde{\Gamma}\}$ in a small tubular neighbourhood of Γ .

Otherwise, after composing with a diffeomorphism of $[-1, \infty)$ we may assume that $f(\text{cl}(\Omega \setminus \Omega_0)) = [0, 1]$. Define $\tilde{f}(x) = -f(x)$. We apply Lemma 4.2 to the restriction $\tilde{f} : M \setminus \Omega \rightarrow (-\infty, 0]$. It follows that there exists a Morse function \tilde{g} , such that $\tilde{g}^{-1}(0) = \Gamma$ and $\mathcal{H}^n(\tilde{g}^{-1}(-t)) \leq \mathcal{H}^n(\partial(f^{-1}([t, \infty)) \setminus \Omega)) + \varepsilon/2$. We define $g(x) = -\tilde{g}(x)$ for $x \in M \setminus \Omega$ and extend it to a Morse function from M to $[-1, \infty)$ in an arbitrary way. By property (2) of Lemma 4.2 we have that (possibly after a small perturbation) $\Omega_1 = g^{-1}([-1, 1)) \supset \Omega_1$.

The bound on the area is similar to the argument in the proof of I. It follows by (2)' that $\mathcal{H}^n(\tilde{G}_t) \leq \mathcal{H}^n(\Gamma_t \setminus \Omega) + \mathcal{H}^n(\Gamma_t \cap \Omega) + \varepsilon/2 < A + \varepsilon$.

If $\Omega \subset \Omega_1$ then by property (4) of Lemma 4.2 we may assume that $\Omega_1 = g^{-1}(1) = \Omega_1$. \square

The second lemma in this section will deal with extending a Morse foliation.

The following result of Falconer ([Fa], see also [Gu1, Appendix 6]) will be used in the proof.

Theorem 5.2. (Falconer) *There exists a constant $C(n)$ so that the following is true. Let $U \subset \mathbb{R}^{n+1}$ be an open set with smooth boundary. There exists a line $l \in \mathbb{R}^{n+1}$, so that projection p_l onto l satisfies $\text{Vol}_n(U \cap p_l^{-1}(t)) < C(n)\text{Vol}_{n+1}(U)^{\frac{n}{n+1}}$ for all $t \in l$. Moreover, we can assume that p_l restricted to ∂U is a Morse function.*

Lemma 5.3. *Let $\varepsilon > 0$, $L > 0$. Suppose $\Omega_0 \subset \Omega_1$ are bounded open sets with smooth boundary and $\Omega_1 \setminus \Omega_0 \subset U$, where U is $(1 + L)$ -bilipschitz diffeomorphic to an open subset of \mathbb{R}^{n+1} . There exists a constant $C(n)$ and a nested family $\{\Gamma'_t\}$ with a family of corresponding open sets $\{\Omega'_t\}$, such that*

$$(1) \mathcal{H}^n(\Gamma'_t) \leq \mathcal{H}^n(\partial \Omega_0) + \mathcal{H}^n(\partial(\Omega_1 \setminus \Omega_0)) + C(n)(1 + L)^n \mathcal{H}^{n+1}(\Omega_1 \setminus \Omega_0)^{\frac{n}{n+1}} + \varepsilon;$$

- (2) Ω'_0 is an inward ε -perturbation of Ω_0 and $\Omega'_1 = \Omega_1$;
 Alternatively, we can require that instead of (2) the family satisfies
 (2') Ω'_1 is an outward ε -perturbation of Ω_1 and $\Omega'_0 = \Omega_0$;

Proof. Let Ω' be an inward $\varepsilon/8$ -perturbation of $\Omega_1 \setminus \Omega_0$. By Theorem 5.2 there exists a Morse function $f : \Omega' \rightarrow [0, 1]$ with fibers of area at most $C(n)(1 + L)^n \mathcal{H}^{n+1}(\Omega_1 \setminus \Omega_0)^{\frac{n}{n+1}} + \varepsilon/4$. By Lemma 4.1 there exists a nested sweepout of Ω' $\{\Sigma_t^a\}$ with a corresponding family of open sets $\{\Xi_t^a\}$, such that $\Xi_1^a = \Omega'$ and $\mathcal{H}^n(\Sigma_t^a) \leq \mathcal{H}^n(\partial(\Omega_1 \setminus \Omega_0)) + C(n)(1 + L)^n \mathcal{H}^{n+1}(\Omega_1 \setminus \Omega_0)^{\frac{n}{n+1}} + \varepsilon/2$.

Let $\{\Sigma_t^b\}$ be a nested family with a corresponding family of open sets $\{\Xi_t^b\}$, such that Ξ_0^b is an inward $\varepsilon/2$ -perturbation of Ω_0 , Ξ_1^b is an inward $\varepsilon/8$ -perturbation of Ω_0 and the areas of all hypersurfaces are at most $\mathcal{H}^n(\partial\Omega) + \varepsilon/2$.

By Lemma 4.3 there exists a nested family $\{\Sigma_t^c\}$ with a corresponding family of open sets $\{\Xi_t^c\}$, such that $\Xi_1^c = \Omega_1$, $\Xi_0^c = \Xi_1^c \sqcup \Xi^2$, where Ξ^1 is an inward $\varepsilon/8$ -perturbation of Ω_0 and Ξ^2 is an inward $\varepsilon/8$ -perturbation of $\Omega_1 \setminus \Omega_0$. It follows from the properties of perturbations that, without any loss of generality, we may assume $\Xi^1 = \Xi_1^b$ and $\Xi^2 = \Omega'$.

We define $\Gamma_t' = \Sigma_{2t}^a \cup \Sigma_{2t}^b$ for $t \in [0, 1/2)$ and $\Gamma_t' = \Sigma_{2t-1}^c$ $t \in [0, 1/2]$ with the open sets defined correspondingly.

We leave it to the reader to verify that a similar construction yields a family satisfying (2') instead of (2). \square

6. NESTED SWEEPOUTS

In this section we prove the following proposition.

Proposition 6.1. *For every $\varepsilon > 0$, given a family of hypersurfaces $\{\Gamma_t\}$ with the corresponding family of open sets $\{\Omega_t\}$ and $\mathcal{H}^n(\Gamma_t) \leq A$, there exists a nested family $\{\tilde{\Gamma}_t\}$ with the corresponding family of open sets $\{\tilde{\Omega}_t\}$, such that $\tilde{\Omega}_0 \subset \Omega_0$, $\Omega_1 \subset \tilde{\Omega}_1$ and $\mathcal{H}^n(\tilde{\Gamma}_t) \leq A + \varepsilon$.*

In particular, for any bounded open set $U \subset M$ with smooth boundary we have $W(U) = W_n(U)$.

The proof proceeds in three steps.

6.1. Step 1. Preliminary modification of the family. We start by replacing the original family $\{\Gamma_t\}$ with a new family $\{\Gamma_t'\}$ that possesses the property that every hypersurface in the family nearly coincides in the complement of a small ball with some hypersurface from a finite list $\{\Gamma_{t_i}'\}$. This construction is inspired by constructions of families, which are continuous in the mass norm in the work of Pitts and Marques-Neves (see [Pi, 4.5] and [MN1, Theorem 14.1]).

Lemma 6.2. *For any $\varepsilon > 0$ there exists a partition $0 = t_0 < \dots < t_N = 1$ of $[0, 1]$ and a family $\{\Gamma'_t\}$ with the corresponding family of open sets $\{\Omega'_t\}$, such that the following holds:*

(1.1) $\Omega'_0 \subset \Omega_0$ and $\Omega_1 \subset \Omega'_1$;

(1.2) $\sup\{\mathcal{H}^n(\Gamma'_t)\} < \sup\{\mathcal{H}^n(\Gamma_t)\} + \varepsilon$;

(1.3) *For each $i = 0, \dots, N-1$ we have one of the two possibilities:*

A. $\Omega'_{t_i} \subset \Omega'_{t_{i+1}}$ and there exists a Morse function $g_i : cl(\Omega_{t_{i+1}} \setminus \Omega_{t_i}) \rightarrow [t_i, t_{i+1}]$, such that $\Gamma'_t = g_i^{-1}(t)$ and $\Omega'_t = \Omega_{t_i} \cup g^{-1}(-\infty, t)$ for $t \in [t_i, t_{i+1}]$.

B. $\Omega'_{t_{i+1}} \subset \Omega'_{t_i}$ and there exists a Morse function $g_i : cl(\Omega_{t_i} \setminus \Omega_{t_{i+1}}) \rightarrow [t_i, t_{i+1}]$, such that $\Gamma'_t = g_i^{-1}(t)$ and $\Omega'_t = \Omega_{t_i} \setminus g^{-1}(-\infty, t]$ for $t \in [t_i, t_{i+1}]$.

Proof. Let M' be a compact subset of M that contains the closure of Ω_t for all $t \in [0, 1]$. Choose r sufficiently small so that for every ball B of radius less or equal to r in M' the following holds:

(i) B is $(1 + \frac{\varepsilon}{100W})^{1/n}$ -bilipschitz diffeomorphic to the Euclidean ball of the same radius;

(ii) $\mathcal{H}^n(B \cap \Gamma_t) < \frac{\varepsilon}{20}$

Let $\{B_i\}$ be a collection of k balls of radius r covering M' , such that balls of half the radius cover M' . We choose a partition $0 = s_0 < \dots < s_{N'} = 1$, such that

(iii) $\mathcal{H}^{n+1}(B_i \cap (\Omega_{s_j} \setminus \Omega_{s_{j+1}})) + \mathcal{H}^{n+1}(B_i \cap (\Omega_{s_{j+1}} \setminus \Omega_{s_j})) < \min\{\frac{r\varepsilon}{10k}, (\frac{\varepsilon}{10})^{\frac{n+1}{n}}\}$

for each $j = 0, \dots, N'$ and $i = 1, \dots, k$.

We define the new family $\{\Gamma'_t\}$ as follows. For $t = s_j$ we set $\Omega'_t = \Omega_t$ and $\Gamma'_t = \partial\Omega_t$, unless Γ_t is a finite collection of points in which case we set $\Gamma'_t = \Gamma_t$ and $\Omega'_t = \emptyset$.

Define a subdivision of $[s_j, s_{j+1}]$ into $2k$ subintervals, $s_j = s_j^0 < \dots < s_j^{2k} = s_{j+1}$. Let $\{B'_i\}$ be a collection of k balls concentric with B_i of radius between $r/2$ and r and such that $\partial B'_i$ intersects Γ_{s_j} and $\Gamma_{s_{j+1}}$ transversally. Set $U_j^1 = \Omega_{s_j} \setminus \Omega_{s_{j+1}}$ and $U_j^2 = \Omega_{s_{j+1}} \setminus \Omega_{s_j}$. By coarea formula and property (iii) for our choice of the subdivision $0 = s_0 < \dots < s_{N'} = 1$ we may assume that B'_i satisfies $\mathcal{H}^n(\partial B'_i \cap (U_j^1 \cup U_j^2)) \leq \frac{\varepsilon}{4k}$.

By our choice of B_i we have that the collection of balls $\{B'_i\}_{i=1}^k$ still cover M' . Inductively we define

$$\begin{aligned}\Omega'_{s_j^0} &= \Omega'_{s_j} \\ \Omega'_{s_j^{2i-1}} &= \Omega'_{s_j^{2i-2}} \setminus (B_i \cap U_j^1) \\ \Omega'_{s_j^{2i}} &= \Omega'_{s_j^{2i-1}} \cup (B_i \cap U_j^2)\end{aligned}$$

for $i = 1, \dots, k$.

Surfaces $\partial\Omega'_{s_j^i}$ may not be smooth, but there exists an arbitrarily small perturbation so that the boundaries are smooth (see Section 3.2). We perform these perturbations in the inward direction for $\Omega'_{s_j^{2i-1}}$ and in the outward direction for $\Omega'_{s_j^{2i}}$.

To simplify notation we do not rename the sets after the perturbations; since the perturbations are arbitrarily small all the estimates for areas and volumes remain valid.

The following properties follow from the definition and (i)-(ii):

- (a) $|\mathcal{H}^n(\partial \Omega'_{s_j}) - \mathcal{H}^n(\Gamma_{s_j})| < \varepsilon/2$;
- (b) $\Omega'_{s_j^{2i-1}} \subset \Omega_{s_j^{2i}}$ and $\Omega_{s_j^{2i-1}} \subset \Omega_{s_j^{2i-2}}$.

We define $\Gamma'_{s_j^l} = \partial \Omega'_{s_j^l}$, unless $\Omega'_{s_j^l}$ is empty. If $\Omega'_{s_j^l}$ is empty we set $\Gamma'_{s_j^l}$ to be a point inside $\Omega'_{s_j^{l-1}}$.

To complete our construction we need to show existence of two types of nested families: a nested family that starts on $\Gamma'_{s_j^{2i-1}}$ and ends on $\Gamma'_{s_j^{2i-2}}$; a nested family that starts on $\Gamma'_{s_j^{2i-1}}$ and ends on $\Gamma'_{s_j^{2i}}$. In both cases we want the homotopies to satisfy the desired upper bound on the areas.

Consider the set $\Omega'_{s_j^{2i-2}} \setminus \Omega'_{s_j^{2i-1}} = B_i \cap U_j^1$. After smoothing the corner (see Section 3.2) we call this set U . We map B_{j+1} to \mathbb{R}^{n+1} by a $(1 + \frac{\varepsilon}{100W})^{1/n}$ -bilipschitz diffeomorphism. Existence of the desired nested families follows by properties (i)-(iii) and Lemma 5.3. \square

6.2. Step 2. Local monotonicization. Assume that family $\{\Gamma_t\}$ satisfies conclusions of Lemma 6.2 for the subdivision $0 = t_0 < \dots < t_N = 1$.

For every $\varepsilon > 0$ and each $i = 0, \dots, N-1$ we will define sets Ω_0^i and Ω_1^i , such that the following holds:

- (2.1) $\Omega_0^i \subset \Omega_1^i$;
- (2.2) $\max\{\mathcal{H}^n(\partial \Omega_0^i), \mathcal{H}^n(\partial \Omega_1^i)\} \leq \max\{\mathcal{H}^n(\Gamma_{t_i}), \mathcal{H}^n(\Gamma_{t_{i+1}})\}$;
- (2.3) $\Omega_{t_{i+1}} \subset \Omega_1^i$ and $\Omega_0^i \subset \Omega_{t_i}$;
- (2.4) There exists a nested family of hypersurfaces $\{\Gamma_t^i\}$, $0 \leq t \leq 1$, with the corresponding family of nested open sets Ω_t^i , such that $\mathcal{H}^n(\Gamma_t^i) \leq \max\{\mathcal{H}^n(\Gamma_{t_i}), \mathcal{H}^n(\Gamma_{t_{i+1}})\} + \varepsilon$.

Definition of Ω_0^i and Ω_1^i

Assume (2.1) - (2.4) are satisfied for all Ω_0^j and Ω_1^j for $j < i$. By Lemma 6.2 (1.3) we only need to consider the following two cases:

(A) $\Omega_{t_i} \subset \Omega_{t_{i+1}}$. In this first case we define $\Omega_0^i = \Omega_{t_i}$ and $\Omega_1^i = \Omega_{t_i}$. Properties (2.1)-(2.3) follow immediately from the definition. Property (2.5) follows by Lemma 6.2 (1.3).

(B) $\Omega_{t_{i+1}} \subset \Omega_{t_i}$. In the second case we consider two subcases:

(B1) Suppose $\mathcal{H}^n(\partial \Omega_{t_i}) \geq \mathcal{H}^n(\partial \Omega_{t_{i+1}})$. We define $\Omega_0^i = \Omega_{t_{i+1}} \setminus cl(N_\delta(\partial \Omega_{t_{i+1}}))$, where $\delta > 0$ is chosen sufficiently small so that $cl(N_\delta(\partial \Omega_{t_{i+1}}))$ is diffeomorphic to

$\partial N \times [-\delta, \delta]$ and hypersurfaces equidistant from ∂N in this neighbourhood all have areas less than $\mathcal{H}^n(\partial \Omega_{t_{i+1}}) + \varepsilon/2$. We set $\Omega_1^i = \Omega_{t_{i+1}}^i$.

(B2) Suppose $\mathcal{H}^n(\partial \Omega_{t_i}) < \mathcal{H}^n(\partial \Omega_{t_{i+1}})$. We set $\Omega_0^i = \Omega_{t_i}^i$ and $\Omega_1^i = N_\delta(\Omega_{t_i})$, where $\delta > 0$ is chosen as in (B1) to guarantee property (2.4).

It is straightforward to verify that with these definitions Ω_0^i and Ω_1^i satisfy (2.1)-(2.4).

The following important property is an immediate consequence of (2.3):

$$(2.5) \quad \Omega_0^{i+1} \subset \Omega_1^i.$$

Informally, the reason why (2.5) holds is because to construct Ω_0^{i+1} we push $\Omega_{t_{i+1}}$ inwards (or not at all) and to construct Ω_1^i we push $\Omega_{t_{i+1}}$ outwards (or not at all).

6.3. Step 3. Gluing two nested families. We prove the following

Proposition 6.3. *Suppose $\{\Gamma_t^a\}$ and $\{\Gamma_t^b\}$ are two nested families (with corresponding families of open sets $\{\Omega_t^a\}$ and $\{\Omega_t^b\}$ respectively) and $\mathcal{H}^n(\Gamma_t^i) \leq W$. Suppose moreover that $\Omega_0^b \subset \Omega_1^a$. For any $\varepsilon > 0$ there exists a nested family $\{\Gamma_t\}$ and a corresponding family of open sets $\{\Omega_t\}$, such that $\mathcal{H}^n(\Gamma_t) \leq W + \varepsilon$, $\Omega_1^b \subset \Omega_1^a$ and $\Omega_0 \subset \Omega_0^a$.*

Proof. The idea for the proof is shown in Figure 5.

Let \mathcal{S} denote the collection of all open sets Ω' , such that $\Omega_0^b \subset \Omega' \subset \Omega_1^a$ and $\partial \Omega'$ is smooth. Let $A = \inf_{\Omega' \in \mathcal{S}} \mathcal{H}^n(\partial \Omega')$ and choose $\Omega \in \mathcal{S}$ with $\mathcal{H}^n(\partial \Omega) < A + \varepsilon/4$. We set $\alpha = \partial \Omega$.

We claim that Ω and α satisfy properties (i) and (ii) from Lemma 5.1(I) for $\Omega_t = \Omega_t^a$. Indeed, if Ω' satisfies $\Omega \subset \Omega' \subset \Omega_1^a$ then $\Omega' \in \mathcal{S}$ and $\mathcal{H}^n(\partial \Omega') < \mathcal{H}^n(\alpha) + \varepsilon/4$. By Lemma 5.1(I) there exists a nested family $\{\tilde{\Gamma}_t^a\}$ with the corresponding family of open sets $\{\tilde{\Omega}_t^a\}$, such that $\tilde{\Omega}_0^a \subset \Omega_0^a$, $\tilde{\Gamma}_1^a = \alpha$ and $\mathcal{H}^n(\tilde{\Gamma}_t^a) \leq W + \varepsilon$.

We claim that Ω and α also satisfy properties (i)' and (ii)' from Lemma 5.1(II) for $\Omega_t = \Omega_t^b$. Indeed, if there is an open set Ω' with $\Omega_0^b \subset \Omega' \subset \Omega$ then again we have $\Omega' \in \mathcal{S}$ and inequality $\mathcal{H}^n(\partial \Omega') < \mathcal{H}^n(\alpha) + \varepsilon/4$ follows by definition of Ω . By Lemma 5.1(II) there exists a nested family $\{\tilde{\Gamma}_t^b\}$ with the corresponding family of open sets $\{\tilde{\Omega}_t^b\}$, such that $\Omega_1^b \subset \tilde{\Omega}_1^b$, $\tilde{\Gamma}_0^b = \alpha$ and $\mathcal{H}^n(\tilde{\Gamma}_t^b) \leq W + \varepsilon$.

We define the desired nested family Γ_t simply by concatenating these two nested families.

□

Now we are ready to complete the proof of Theorem 6.1. We apply local monotization to define families $\{\Gamma_t^i\}$ for $i = 1, \dots, N - 1$.

By (2.5) we have $\Omega_0^2 \subset \Omega_1^1$. Hence, we can apply Proposition 6.3 to the nested families $\{\Gamma_t^1\}$ and $\{\Gamma_t^2\}$. We obtain a new nested family $\Gamma_t^{1,2}$ with the corresponding family of open sets $\{\Omega_t^{1,2}\}$. By (2.3) and Proposition 6.3 we have $\Omega_0^{1,2} \subset \Omega_0^1 \subset \Omega_0$

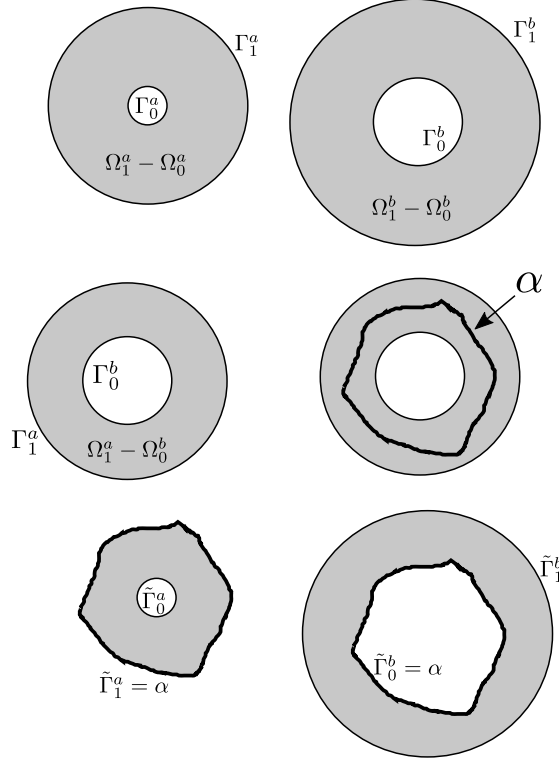


FIGURE 5. Gluing two nested sweepouts.

and $\Omega_{t_2} \subset \Omega_1^2 \subset \Omega_1^{1,2}$. Using (2.5) again we have $\Omega_0^3 \subset \Omega_1^{1,2}$. Hence, we can apply Proposition 6.3 to $\{\Gamma_t^{1,2}\}$ and $\{\Gamma_t^3\}$. We iterate this procedure. At the i -th step we apply Proposition 6.3 to families $\{\Gamma_t^{1,\dots,i}\}$ and $\{\Gamma_t^{i+1}\}$ to construct a new nested family $\{\Gamma_t^{1,\dots,i,i+1}\}$ with $\Omega_0^{1,\dots,i} \subset \Omega_0$ and $\Omega_1 \subset \Omega_1^{1,\dots,i}$. Proposition 6.3 and (2.5) guarantee that $\Omega_0^{i+2} \subset \Omega_1^{1,\dots,i}$, so we can go to the next step.

After performing this operation N times we obtain the desired nested family. This finishes the proof of Theorem 6.1.

7. NO ESCAPE TO INFINITY

In this section we prove Proposition 2.1, which we recall below.

Proposition 2.1 *For every good set U there exists a positive constant $\varepsilon(U)$ which depends only on U such that the following holds. For every good sweepout $\{\Gamma_t\}$ of U with associated family of open sets $\{\Omega_t\}$, there is a surface Γ_{t^*} in the collection which has area at least $W_g(U)$, and such that $\mathcal{H}^n(\Gamma_{t^*} \cap \text{cl}(U)) \geq \varepsilon(U)$.*

The proof is by contradiction. We assume that Proposition 2.1 does not hold and construct a good sweepout with volume of hypersurfaces strictly less than $W_g(U)$. The main tool in the proof is Theorem 6.1.

Let U be a good set.

Lemma 7.1. *There exists $\varepsilon(U) > 0$, $\varepsilon_0(U) > 0$ and $\varepsilon_1(U) > 0$ such that for any open set Ω' the following holds:*

- (1) $\max\{\varepsilon, \varepsilon_1\} < \mathcal{H}^n(\partial U)/10$.
- (2) If $\varepsilon_0 < \mathcal{H}^{n+1}(\Omega' \cap U) < \mathcal{H}^{n+1}(U) - \varepsilon_0$ then $\mathcal{H}^n(\partial \Omega' \cap U) > 2\varepsilon$.
- (3) A) If $\mathcal{H}^{n+1}(\Omega' \cap U) < 2\varepsilon_0$ then there exists a family of open sets $\{\Xi_t\}$ with $\Xi_0 = \Omega'$, $\Xi_t \setminus N_{\varepsilon_1}(U) = \Omega' \setminus N_{\varepsilon_1}(U)$, $\Xi_1 \cap U = \emptyset$ and $\mathcal{H}^n(\partial \Xi_t) < \mathcal{H}^n(\partial \Omega') + \varepsilon_1$.
 B) If $\mathcal{H}^{n+1}(\Omega' \cap U) > \mathcal{H}^{n+1}(U) - 2\varepsilon_0$ then there exists a family of open sets $\{\Xi_t\}$ with $\Xi_0 = \Omega'$, $\Xi_t \setminus N_{\varepsilon_1}(U) = \Omega' \setminus N_{\varepsilon_1}(U)$, $\Xi_1 \cap U = U$ and $\mathcal{H}^n(\partial \Xi_t) < \mathcal{H}^n(\partial \Omega') + \varepsilon_1$.

Proof. Pick any $\varepsilon_1 \in (0, \mathcal{H}^n(\partial U)/10)$. We will show that for all sufficiently small ε_0 (with the choice of ε_0 depending on ε_1) statement (3) holds; we will show that for all sufficiently small ε (with the choice of ε depending on ε_0) statement (2) holds.

Statement (2) follows from the properties of the isoperimetric profile of $cl(U)$.

Now we will prove Statement (3) A). Statement (3) B) follows by an analogous argument. The argument is similar to the proof of Proposition 4.3 in [GL] (see also Lemma 7.1 in [Mo]).

Let $r_0 > 0$ be sufficiently small, so that every ball B of radius $r \in (0, r_0]$ centered at a point in U is 2-bilipschitz diffeomorphic to a ball of the same radius in the Euclidean space.

Choose a covering $\{B_i\}$ of U by balls of radius r_0 , so that concentric balls of radius $\frac{r_0}{4}$, denoted by $\frac{1}{4}B_i$, still cover U . Using coarea formula we may choose a covering $\{B'_i\}$ of U by N balls of radius $r_i \in (r_0/2, r_0)$, so that $\mathcal{H}^n(\partial B'_i \cap \Omega') \leq \frac{4\varepsilon_0}{r_0}$.

Given an $(n-1)$ -dimensional compact submanifold $\gamma \subset B'_i$ we say that an n -dimensional manifold (with boundary) $\Sigma \subset B'_i$ is a δ -minimizing filling of γ if $\partial \Sigma = \gamma$ and for every other submanifold Σ' filling γ in $cl(B'_i)$ we have $\mathcal{H}^n(\Sigma) \leq \mathcal{H}^n(\Sigma') + \delta$.

By Lemma 4.6 in [GL] there exists a constant $c_0(n)$, so that if A is an open set in $\partial B'_i$ with $\mathcal{H}^n(A) \leq c_0(n)r_0^n$ then for every $\delta > 0$ there exists a δ -minimizing filling Σ of ∂A in B'_i , so that Σ does not intersect $\frac{1}{4}B_i$.

Set $\delta = \frac{\varepsilon_1}{10N}$ and $\varepsilon_0 = \min\{\frac{\varepsilon_1 r_0}{40}, \frac{c_0 r_0^{n+1}}{4}, (\frac{\varepsilon_1}{10})^{\frac{n+1}{n}}\}$. We will inductively remove Ω' from each $\frac{1}{4}B_i$. Since $\{\frac{1}{4}B_i\}$ cover U the desired conclusion follows.

Start with B'_1 . First we use Lemma 5.3 to construct a nested family that starts on $\partial \Omega'$ and ends on the smoothing of $\partial \Omega' \setminus B'_i$. Let Σ_i be a δ -minimizing filling for $\partial(\Omega' \cap \partial B'_i)$, which does not intersect $\frac{1}{4}B_1$. Note that by definition of δ -minimizing we have $\mathcal{H}^n(\Sigma_i) \leq \mathcal{H}^n(\partial \Omega' \cap B'_i) + \delta$. The second step is to construct a family that starts on a smoothing of $\partial \Omega' \setminus B'_i$ and ends on a smoothing of $(\partial \Omega' \setminus B'_i) \cup \Sigma_i$. Note

that during these two deformations the areas of hypersurfaces are bounded above by $\mathcal{H}^n(\partial\Omega') + \varepsilon_1$ and in the end of the second step the area of the hypersurface is bounded above by $\mathcal{H}^n(\partial\Omega') + \frac{\varepsilon_1}{10N}$.

We iterate this procedure for each ball B'_i . Since at the end of the deformation in each ball we only accumulate an increase in area of at most $\frac{\varepsilon_1}{10N}$ the total increase in area will be below ε_1 . □

Proof of Proposition 2.1. Suppose Proposition 2.1 does not hold. Then there exists a good sweepout $\{\Gamma_t\}_{t \in [0,1]}$, such that if $\mathcal{H}^n(\Gamma_t) \geq W_g(U)$ then $\mathcal{H}^n(\Gamma_t \cap U) < \varepsilon(U)$. Let $\{\Omega_t\}$ denote the corresponding family of open sets. Let $f(t) = \mathcal{H}^n(\Gamma_t \cap U)$. Note that $f(t)$ may not be continuous. However, it is easy to see that one can perturb the family $\{\Gamma_t\}$ so that it is roughly continuous in the following sense.

Definition 7.2. Function $f(t)$ is δ -continuous if the oscillation $\omega_f(t) = \lim_{a \rightarrow 0} [\sup_{s \in [t-a, t+a]} f(s) - \inf_{s \in [t-a, t+a]} f(s)]$ satisfies $\omega_f(t) < \delta$ for every t .

Lemma 7.3. Let U be a bounded open set with smooth boundary and $\{\Gamma_t\}$ be a good sweepout of U . For every $\delta > 0$ there exists a good sweepout $\{\Gamma'_t\}$ of U , such that $f(t) = \mathcal{H}^n(\Gamma'_t \cap U)$ is δ -continuous, $\sup_t \mathcal{H}^n(\Gamma'_t) \leq \sup_t \mathcal{H}^n(\Gamma_t) + \delta$ and $\sup_t \mathcal{H}^n(\Gamma'_t \cap U) \leq \sup_t \mathcal{H}^n(\Gamma_t \cap U) + \delta$.

Proof. This follows from the construction in the proof of Lemma 6.2. □

Hence, without any loss of generality we may assume that sweepout $\{\Gamma_t\}$ satisfies the conclusions of Lemma 7.3 for $\delta < \varepsilon/10$ and that for all Γ_t with $\mathcal{H}^n(\Gamma_t) \geq W_g(U)$ we have $\mathcal{H}^n(\Gamma_t \cap U) < 1.1\varepsilon(U)$.

Let $g : [0, 1] \rightarrow [0, \mathcal{H}^{n+1}(U)]$ be defined as $g(t) = \mathcal{H}^{n+1}(U \cap \Omega_t)$. Function $g(t)$ is continuous. By Lemma 7.1 (2) each connected component I' of $g^{-1}([\varepsilon_0, \mathcal{H}^{n+1}(U) - \varepsilon_0])$ is contained in some interval $I = [t_0, t_1] \subset [0, 1]$, such that $f(t) \geq \frac{3}{2}\varepsilon$ for all $t \in I$. Moreover, by Lemma 7.3 we may assume that $\varepsilon \leq f(t_i) \leq 2\varepsilon$, $i = 0, 1$. By continuity of $g(t)$ and since $\{\Gamma_t\}$ is a sweepout there exists an interval I as above with $\mathcal{H}^{n+1}(\Omega_{t_0} \cap U) \leq \varepsilon_0$ and $\mathcal{H}^{n+1}(\Omega_{t_1} \cap U) \geq \mathcal{H}^{n+1}(U) - \varepsilon_0$.

By construction we have that $\mathcal{H}^n(\Gamma_t) < W_g(U) - \delta$ for some $\delta > 0$ and for all $t \in I$.

We would like to turn $\{\Gamma_t\}$ into a good sweepout of U , while retaining an upper bound on the volume below $W_g(U)$. The family $\{\Gamma_t\}_{t \in I}$ fails to be a good sweepout of U for two reasons:

1. $\Omega_{t_0} \cap U$ and $\Omega_{t_1} \setminus U$ are not empty;
2. $\mathcal{H}^n(\Gamma_{t_0})$ and $\mathcal{H}^n(\Gamma_{t_1})$ may be larger than $5\mathcal{H}^n(\partial U)$. In fact, they may be as large as the largest hypersurface in $\{\Gamma_t\}_{t \in I}$.

To address the first problem we note that $\Omega_{t_0} \cap U$ and $\Omega_{t_1} \setminus U$ have volume at most ε_0 and we may use Lemma 7.1 to homotope Γ_{t_0} and Γ_{t_1} outside of U while increasing

the \mathcal{H}^n -measure of the hypersurfaces by a controlled amount. Observe, however, that if δ is much smaller than ε and $\mathcal{H}^n(\Gamma_{t_i})$ is almost equal to $W_g(U) - \delta$ then the resulting family will have volume larger than $W_g(U)$. The second problem seems even more substantial.

The main tool to resolve these two problems is to replace $\{\Gamma_t\}_{t \in I}$ with a nested family. This allows us to define certain two nearly volume minimizing hypersurfaces. We then modify the nested family so that it starts and ends on these two hypersurfaces, which have small area and can be “homotoped” away from U to produce a good sweepout.

We apply Theorem 6.1 to construct a nested family $\{\bar{\Gamma}_t\}$, $t \in [0, 1]$, such that $\mathcal{H}^n(\bar{\Gamma}_t) < W_g(U) - \frac{\delta}{2}$, $\bar{\Omega}_0 \subset \Omega_{t_0}$ and $\Omega_{t_1} \subset \bar{\Omega}_1$.

The situation is depicted on Figure 6. It will be useful to define the set $P = (\Omega_{t_0} \cap U) \cup (U \setminus cl(\Omega_{t_1}))$. P will play an important role for three reasons:

$$(2.1) \quad \mathcal{H}^{n+1}(P) < 2\varepsilon_0$$

$$(2.2) \quad \mathcal{H}^n(\partial P \cap U) \leq 4\varepsilon$$

$$(2.3) \quad \bar{\Omega}_0 \cap U \text{ and } (M \setminus cl(\bar{\Omega}_1)) \cap U \text{ are contained in } cl(P).$$

Lemma 7.4. *There exists $t' \in [0, 1]$, such that $\mathcal{H}^n(\bar{\Gamma}_{t'} \setminus U) \leq 2\mathcal{H}^n(\partial U)$.*

Proof. Let $L = \max_t \{\mathcal{H}^n(\bar{\Gamma}_t \cap U)\}$. Let \bar{U} denote an inward δ -perturbation of $U \setminus cl(P)$. We have that $\{\Gamma_t\}$ is a nested sweepout of \bar{U} . By Lemma 4.1 there exists a nested sweepout of \bar{U} by hypersurfaces of area at most

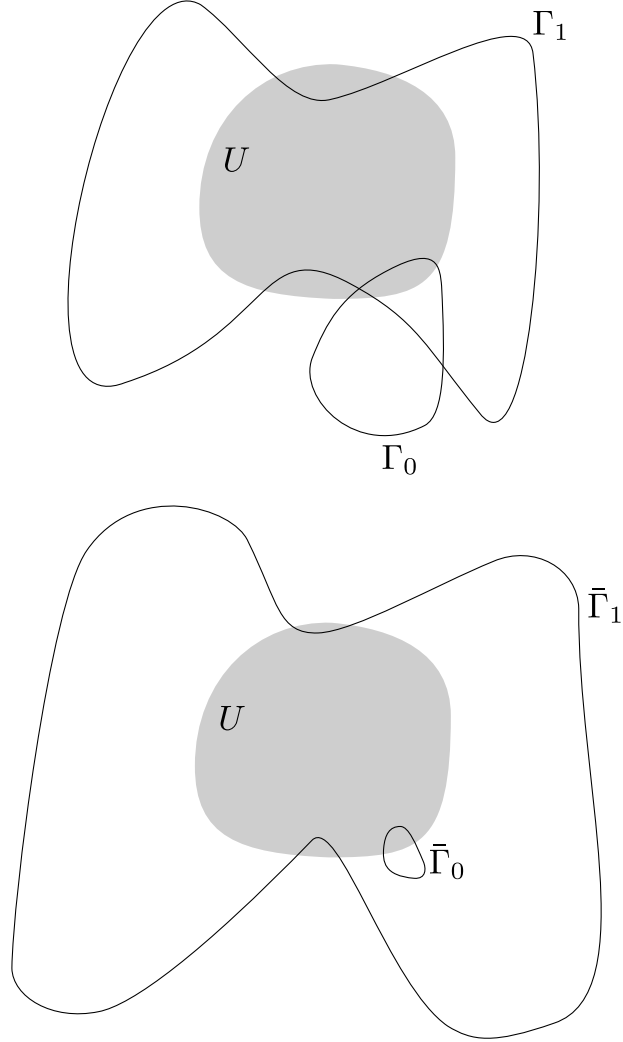
$$\begin{aligned} L + \mathcal{H}^n(\partial \bar{U}) + \delta &\leq L + \mathcal{H}^n(\partial U) + 4\varepsilon + 2\delta \\ &\leq L + 2\mathcal{H}^n(\partial U) \end{aligned}$$

Moreover, this sweepout starts on a hypersurface of area 0 and ends on $\partial \bar{U}$. By Lemma 7.1 we can deform $\partial \bar{U}$ outside of U through hypersurfaces of controlled area.

We have produced a good sweepout of U with maximal volume of the hypersurface at most $L + 2\mathcal{H}^n(\partial U)$. By definition of W_g we have $L + 2\mathcal{H}^n(\partial U) \leq W_g$. Hence, $\mathcal{H}^n(\bar{\Gamma}_{t'}) < W_g(U)$ implies that for some $t' \in [0, 1]$ we have $\mathcal{H}^n(\bar{\Gamma}_{t'} \setminus U) < 2\mathcal{H}^n(\partial U)$. \square

We will construct a good sweepout of \bar{U} with hypersurfaces of area at most $W_g(U) - \delta$, starting and ending on hypersurfaces less than $5\mathcal{H}^n(\partial U)$. By Lemma 7.1 we can deform it into a good sweepout of U by hypersurfaces of area at most $W_g(U) - \delta/4$. This contradicts the definition of $W_g(U)$ and so Proposition 2.1 follows.

To construct a good sweepout of \bar{U} with these properties we proceed as follows. Let t' be as in Lemma 7.4, and let \mathcal{U}_0 denote a collection of all open sets Ω with smooth boundary, such that $\bar{\Omega}_0 \subset \Omega \subset \bar{\Omega}_{t'} \setminus \bar{U}$, where \bar{U} denotes an inward $\frac{\delta}{100}$ -perturbation of $U \setminus cl(P)$. Let \mathcal{U}_1 denote a collection of all open sets Ω with smooth boundary,

FIGURE 6. Replacing family $\{\Gamma_t\}_{t \in I}$ with a nested family $\{\bar{\Gamma}_t\}$

such that $\bar{\Omega}_{t'} \cup \bar{U} \subset \Omega \subset \bar{\Omega}_1$. Let $A_i = \inf\{\mathcal{H}^n(\partial\Omega) : \Omega \in \mathcal{U}_i\}$. Observe that a perturbation of $\bar{\Omega}_{t'} \setminus cl(\bar{U})$ is an element of \mathcal{U}_0 and a perturbation of $\bar{\Omega}_{t'} \cup \bar{U}$ is an element of \mathcal{U}_1 . By Lemma 7.4 the boundary areas of these hypersurfaces are at most $3\mathcal{H}^n(\partial U)$. Hence, it follows from Lemma 7.1 that $A_i \leq 3\mathcal{H}^n(\partial U)$. Let $\Sigma_0 = \partial\Xi_0$ and $\Sigma_1 = \partial\Xi_1$ be two hypersurfaces with $\Xi_i \in \mathcal{U}_i$ and $\mathcal{H}^n(\Sigma_i) \leq A_i + \delta/4$. We have that Ξ_0 is contained in $\bar{\Omega}_{t'}$, and that \bar{U} is contained in its complement, and we also have that Ξ_1 contains both \bar{U} and $\bar{\Omega}_{t'}$. In particular, the set $\Xi_1 \setminus \Xi_0$ contains \bar{U} .

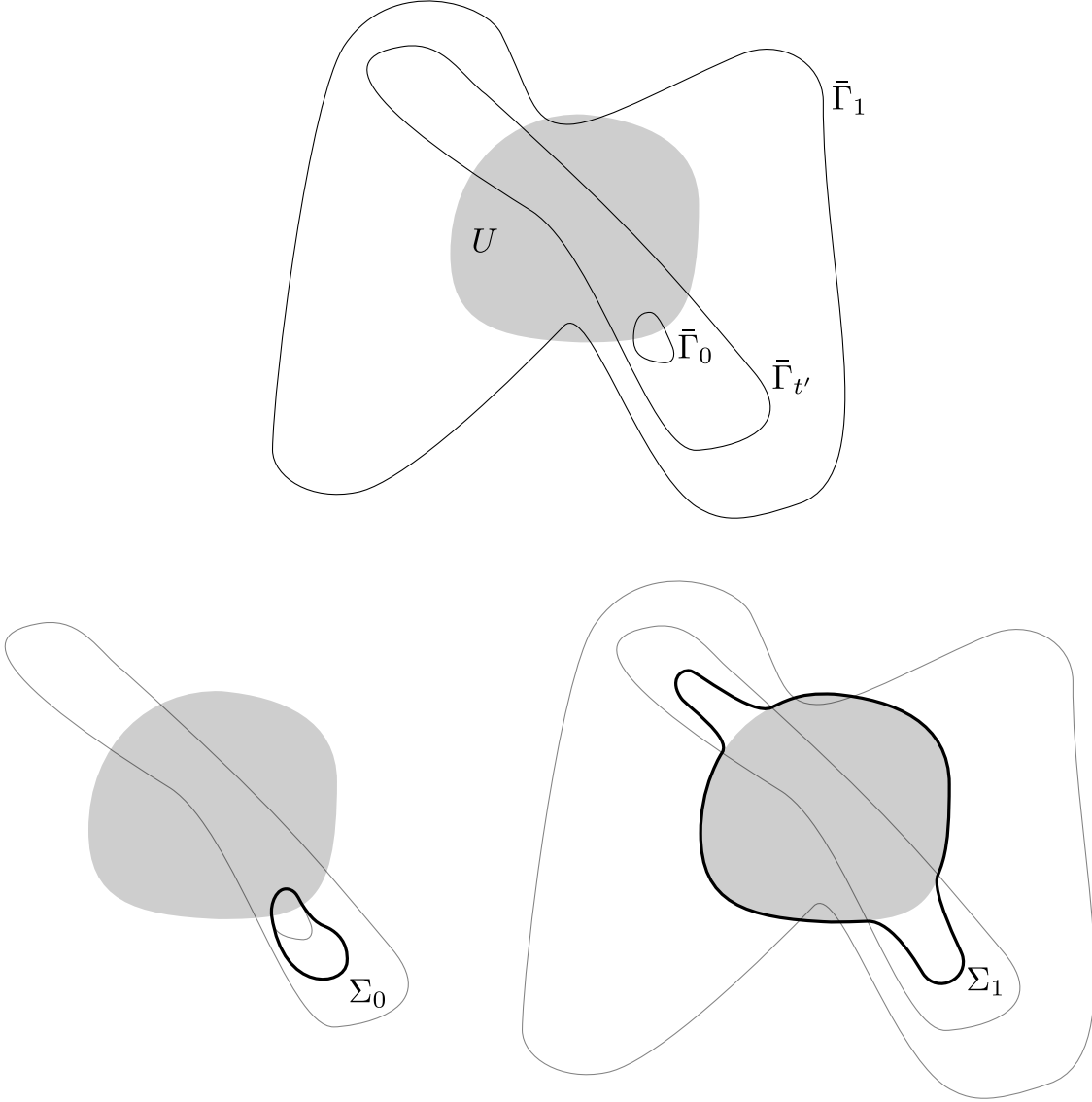


FIGURE 7. Constructing a good sweepout in the proof of Proposition 2.1.

We apply Lemma 5.1 I to construct a nested sweepout of \bar{U} that starts on Σ_0 and ends on $\bar{\Omega}_1$ and is composed of hypersurfaces of area at most $W_g(U) - 3\delta/4$. Here we are using the fact that Ξ_0 is contained in $\bar{\Omega}_1$. We then apply Lemma 5.1 II to this sweepout to produce a nested sweepout of \bar{U} that starts on Σ_0 and ends on Σ_1 and is composed of hypersurfaces of area at most $W_g(U) - \delta/4$. Here we are using

the fact that $Xi_0 \subset Xi_1$. This finishes the proof of Proposition 2.1. This proof is shown in Figure 7.

8. CONVERGENCE OF A MIN-MAX SEQUENCE TO A MINIMAL HYPERSURFACE

8.1. Manifolds with sublinear volume growth. In this section we prove Theorem 1.1 and Corollary 1.3. Corollary 1.3 follows from the following lemma. We show that if M has sublinear volume growth (in particular, if it has finite volume) then it contains a good set.

Lemma 8.1. *Let M^{n+1} be a complete non-compact manifold with sublinear volume growth. There exists a good set $U \subset M$, such that $0 < W_g(U) < \infty$.*

Proof. Let x be such that $\liminf_{r \rightarrow \infty} \frac{\text{Vol}(B_r(x))}{r} = 0$. Fix a small geodesic ball $B_r(x)$ and define an isoperimetric constant $C_I = \inf\{\mathcal{H}^n(\Sigma)\}$, where the infimum is taken over all hypersurfaces in $B_r(x)$, subdividing $B_r(x)$ into two subsets of equal volume. By the coarea formula we can find $R > r$ with $\mathcal{H}^n(\partial B_R(x)) < \frac{C_I}{100}$ and $\partial B_R(x)$ smooth.

It follows that $B_R(x)$ is a good set. The distance function $d_x(y) = \text{dist}(x, y)$ may not be smooth, but there exists a smoothing of this function \tilde{d}_x (see [GW]), such that $\tilde{d}_x = d_x$ in $B_R(x)$ and $|\nabla \tilde{d}_x| \leq 1 + \varepsilon$ for all y . Moreover, we may assume that \tilde{d}_x is a Morse function.

Hence, the set of good sweepouts of $B_R(x)$ is non-empty. Every sweepout of $B_R(x)$ is also a sweepout of $B_r(x)$, so it must contain a hypersurface of area at least C_I . \square

8.2. Proof of Theorem 1.1. Theorem 1.1 follows immediately from the following Theorem.

Theorem 8.2. *Let M^{n+1} be a complete Riemannian manifold of dimension $n + 1$. Suppose M contains a good set U . For every $\delta > 0$ there exists a complete embedded minimal hypersurface Γ , satisfying the following properties:*

- (1) $\mathcal{H}^n(\Gamma) \leq W_\partial(U) + \mathcal{H}^n(\partial U)$;
- (2) $\mathcal{H}^n(\Gamma \cap N_\delta(U)) \geq \frac{\varepsilon(U)}{2}$,

where $\varepsilon(U)$ is as in Lemma 7.1. The hypersurface is smooth in the complement of a closed set of dimension $n - 7$.

Remark 8.3. a) The min-max argument applied to families of good sweepout of a good set U may produce a non-compact minimal hypersurface. Consider the following example. Let S_r denote spheres of radius r in \mathbb{R}^3 . We modify the Euclidean metric on \mathbb{R}^3 , so that the new metric is invariant under rotations around 0, and so that the areas of S_r and lengths of great circles on S_r decay exponentially for $r > 1$.

If the decay is fast enough the min-max argument for good sweepouts of the ball $B_2(0)$ will produce a hyperplane passing through 0 (of area $\pi + \varepsilon$).

b) If U is conformally equivalent to a metric of non-negative Ricci curvature then from [GL] we obtain an upper bound for the volume of the minimal hypersurface $\mathcal{H}^n(\Gamma) \leq C(n) \mathcal{H}^{n+1}(U)^{\frac{n}{n+1}}$.

To prove Theorem 8.2 we use Proposition 2.1 and arguments from [DT]. For the most part in this section we closely follow [DT]. However, some modifications are necessary in construction of the pull-tight deformation and construction of a min-max sequence, which is almost minimizing in all sufficiently small annuli.

The regularity of a stationary varifold obtained from a min-max sequence is proved using the notion of ε -almost minimizing hypersurfaces introduced in [Pi]. We will use the notion of almost minimality from [DT, 2.2].

Definition 8.4. Let $\varepsilon > 0$ and $U \subset M$ open. A boundary $\partial\Omega$ is called ε -almost minimizing in U if there is NO 1-parameter family of boundaries $\{\partial\Omega_t\}$, $t \in [0, 1]$, satisfying the following properties:

- (s1), (s2), (s3), (sw1), and (sw3) of Definition 3.2 hold;
- $\partial\Omega_0 = \Omega$ and $\partial\Omega_t \setminus U = \partial\Omega \setminus U$ for every t ;
- $\mathcal{H}^n(\partial\Omega_t) \leq \mathcal{H}^n(\partial\Omega) + \frac{1}{8}\varepsilon$;
- $\mathcal{H}^n(\partial\Omega_1) \leq \mathcal{H}^n(\partial\Omega) - \varepsilon$

A sequence $\{\partial\Omega_k\}$ of hypersurfaces is called almost minimizing in U if each $\partial\Omega_k$ is ε_k -almost minimizing in U for some sequence $\varepsilon_k \rightarrow 0$.

Let $\mathcal{AN}_r(x)$ denote the set of all open annuli $An(x, t_1, t_2) = B_{t_2}(x) \setminus cl(B_{t_1}(x))$ for $t_1 < t_2 < r$. We have the following result from [DT]:

Proposition 8.5. Let $r : M \rightarrow \mathbb{R}_+$ be a function and $\{\Gamma^k\}$ is a sequence of hypersurfaces, s.t.

- (A) $\{\Gamma^k\}$ is a.m. in every $An(x) \in \mathcal{AN}_{r(x)}(x)$;
- (B) Γ^k converges to a stationary varifold V as $k \rightarrow \infty$.

Then V is induced by an embedded minimal hypersurface, which is smooth on the complement of a closed set of Hausdorff dimension at most $n - 7$.

Proof. This proposition is contained in Propositions 2.6, 2.7 and 2.8 of [DT]. All arguments there are local and therefore they apply to the non-compact case. \square

Proposition 8.6. Let $U \subset M$ be a good set and suppose $W_g(U) < \infty$. For every $\delta > 0$ there exists a function $r : M \rightarrow \mathbb{R}_+$, $\varepsilon > 0$ and a sequence $\{\Gamma^k\}$, such that (A) and (B) of Proposition 8.5 hold and

- (C) $\mathcal{H}^n(\Gamma^k \cap N_\delta(U)) > \varepsilon/2$ for every k .

Remark 8.7. The statement of the proposition remains true if we replace 1-neighbourhood of U with $N_{r_0}(U)$ for any positive r_0 . The function $r : M \rightarrow \mathbb{R}_+$ may change depending on r_0 .

Combining Proposition 8.5 and 8.6 we obtain that M contains a stationary varifold V induced by a minimal hypersurface Σ with $\mathcal{H}^n(\Sigma \cap N_\delta(U)) > \varepsilon/2$. In particular, the intersection of Σ with $N_\delta(U)$ is non-empty and the minimal hypersurface has volume at least $\varepsilon/2$. This implies Theorem 8.2.

The rest of this section will be devoted to the proof of Proposition 8.6.

8.3. Pull-tight. Using terminology from [DT] we say that a sequence $\{\Gamma_t^i\}$ of good sweepouts of U is minimizing if $\lim_{i \rightarrow \infty} \sup_t \mathcal{H}^n(\Gamma_t^i) = W_g(U)$ and a sequence of hypersurfaces $\{\Gamma_{t_i}^i\}$ with $\lim_{i \rightarrow \infty} \mathcal{H}^n(\Gamma_{t_i}^i) \rightarrow W_g(U)$ will be called a min-max sequence.

Let \mathcal{V} denote the space of varifolds in M with mass bounded by $2W_g(U)$. \mathcal{V} is endowed with weak* topology. By the Riesz Representation Theorem and the Banach-Alaoglu Theorem this space is compact and metrizable. Let \mathfrak{d} denote a metric on \mathcal{V} which induces this topology.

Another important metric on the space of varifolds is given by (see [Pi, 2.1(19)])

$$\mathbf{F}(V_1, V_2) = \sup\{V_1(f) - V_2(f) \mid f \in \mathcal{K}(Gr_n(M)), |f| \leq 1, Lip(f) \leq 1\}$$

where $\mathcal{K}(Gr_n(M))$ denotes the set of Lipschitz functions compactly supported in $Gr_n(M)$.

When manifold M is compact the topology of the \mathbf{F} metric and the weak* topology on \mathcal{V} coincide. When M is not compact these topologies are different. Moreover, in this case \mathcal{V} is not compact in the \mathbf{F} metric. The standard pull-tight argument (see [Pi, Theorem 4.3], [CD, Proposition 4.1] and [MN1, Proposition 8.5]) uses compactness with the \mathbf{F} metric in an important way, so in our case the argument has to be modified.

Let $\mathcal{V}_{st} \subset \mathcal{V}$ denote the closed subset of stationary varifolds in \mathcal{V} (see [Si, 8.2]). If Γ is a hypersurface we will slightly abuse notation and write Γ to denote the varifold induced by Γ .

Lemma 8.8. *There exists a minimizing sequence $\{\{\Gamma_t^i\}\}$ of good sweepouts of U , such that for every min-max sequence $\{\Gamma_{t_i}^i\}$ we have $\lim_{i \rightarrow \infty} \mathfrak{d}(\Gamma_{t_i}^i, \mathcal{V}_s) = 0$.*

Let $\Omega \subset M$ be an open subset. Let \mathcal{V}_Ω denote the space of varifolds in Ω with mass bounded by $2W_g(U)$. For varifolds in Ω we can define metric

$$\mathbf{F}_\Omega(V_1, V_2) = \sup\{V_1(f) - V_2(f) \mid f \in \mathcal{K}(Gr_n(\Omega)), |f| \leq 1, Lip f \leq 1\}$$

It follows from the definition that

$$\mathbf{F}_{\Omega_1}(V_1 \llcorner Gr_n(\Omega_1), V_2 \llcorner Gr_n(\Omega_1)) \leq \mathbf{F}_{\Omega_2}(V_1 \llcorner Gr_n(\Omega_2), V_2 \llcorner Gr_n(\Omega_2))$$

whenever $\Omega_1 \subset \Omega_2$. When Ω is a bounded subset of M the weak topology on \mathcal{V}_Ω and the topology induced by the \mathbf{F}_Ω metric coincide.

We will also need the following notation. Let $\mathcal{V}_{\Omega, st}$ denote the set of all stationary varifolds of mass at most $2W_g$ and supported in $Gr_n(\Omega)$.

Lemma 8.9. *Let Ω be a bounded open set. There exists a map $\Phi_\Omega : \mathcal{V} \rightarrow \mathcal{V}$ and monotone sequences of positive numbers $\tau_1 \geq \tau_2 \geq \dots \tau_k \rightarrow 0$ and $\varepsilon_1 \geq \varepsilon_2 \geq \dots \varepsilon_k \rightarrow 0$ with the following properties.*

- (1) $\|\Phi_\Omega(V)\|(M) \leq \|V\|(M)$
- (2) If $\|V\|(M) \leq 5\mathcal{H}^n(\partial U)$ then $\Phi_\Omega(V) = V$
- (3) If $\|V\|(M) \geq 9\mathcal{H}^n(\partial U)$ and $\mathbf{F}_\Omega(V \llcorner Gr_n(\Omega), \mathcal{V}_{\Omega, st}) \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$ then the following holds:
 - A. $\|\Phi(V)\|(M) \leq \|V\|(M) - \varepsilon_k$
 - B. $\mathbf{F}(V, \Phi_\Omega(V)) \leq \tau_k$

Moreover, if $\{\text{support}(V_t)\}$ is a family of hypersurfaces in a sense of Definition 3.1 then so is $\{\text{support}(\Phi_\Omega(V_t))\}$.

Proof. Fix integer $k > 0$. Let $\mathcal{V}'_{\Omega, k}$ be the set of varifolds $V \in \mathcal{V}$ satisfying the following properties:

- 1. $\|V\|(M) \in [9\mathcal{H}^n(\partial U), 2W_g(U)]$;
- 2. $\mathbf{F}_\Omega(V \llcorner Gr_n(\Omega), \mathcal{V}_{\Omega, st}) \in [\frac{1}{2^{k+1}}, \frac{1}{2^k}]$

Let $p(V) = V \llcorner Gr_n(\Omega)$ denote the restriction function and let $\mathcal{V}_{\Omega, k} = p(\mathcal{V}'_{\Omega, k})$. It is straightforward to check that $\mathcal{V}_{\Omega, k}$ is compact in the topology induced by the \mathbf{F}_Ω metric.

We will say that a smooth vector field χ is admissible if χ is compactly supported in Ω , $|\chi|_{C^1} \leq 1$ and $|\chi(x)| \leq \text{dist}(x, \partial\Omega)$. Let X_Ω denote the set of all admissible vector fields. We claim that there exists a $c_k > 0$, such that $\sup_{V \in \mathcal{V}_{\Omega, k}} \inf_{\chi \in X_\Omega} \{\delta V(\chi)\} < -c_k$ for otherwise there would exist a sequence of varifolds $V_i \in \mathcal{V}_{\Omega, k}$ converging (in \mathbf{F}_Ω) to a stationary varifold supported in $Gr_n(\Omega)$, which contradicts condition 2 above. Here, $\delta V(\chi)$ means the first variation of V with respect to the vector field χ .

By compactness (cf. arguments in [Pi, Theorem 4.3], [CD, Proposition 4.1] and [MN1, Proposition 8.5]) we can find a locally finite open covering $\{U_i^k\}$ of $\mathcal{V}_{\Omega, k}$ and a collection of admissible vector fields $\{\chi_i^k\}$, such that $\delta V(\chi) < -\frac{c_k}{2}$ and $U_{i_1}^{k_1}$ is disjoint from $U_{i_2}^{k_2}$ whenever $|k_1 - k_2| \geq 2$. Let $\bar{U}_i^k = p^{-1}(U_i^k) \cap \{V : \|V\|(M) > 6\mathcal{H}^n(\partial U)\}$. We have that the collection $\{\bar{U}_i^k\}$ covers $\mathcal{V}'_{\Omega, k}$ and the union $\bigcup \bar{U}_i^k$ is disjoint from the set of varifolds V with $\|V\|(M) \leq 5\mathcal{H}^n(\partial U)$. Choose a partition of unity $\{\phi_i^k\}$ subordinate to $\{\bar{U}_i^k\}$. Define a continuous family of vector fields $\chi_V = \sum \phi_i^k(V) \chi_i^k$.

We have that χ_V is admissible for all $V \in \mathcal{V}$ and $\delta V(\chi_V) < -\min \frac{1}{2} \{c_{k-1}, c_k, c_{k+1}\}$. Hence, for each χ_V we can define a 1-parameter family of diffeomorphisms $\Psi_V : [0, \infty) \times M \rightarrow M$ with $\frac{\partial \Psi_V(t, x)}{\partial t} = \chi_V(\Psi_V(t, x))$. By definition of admissible vector

field we have that Ψ_V is the identity on $Gr_n(M \setminus \Omega)$. It follows that there exists a continuous choice of $t = t(V)$ and $\varepsilon_k > 0$ so that $\|\Psi_{V\#}(t_V, V)\|(M) \leq \|V\|(M) - \varepsilon_k$ for all $V \in \mathcal{V}'_{\Omega,k}$. Moreover, we may assume that $t_V \leq 1/k$ if $V \in \mathcal{V}'_{\Omega,k}$. We define $\Phi(V) = \Psi_{V\#}(t_V, V)$. Properties A and B follow by construction. \square

We use Lemma 8.9 to prove Lemma 8.8. Let $U_0 \subset U_1 \subset \dots$ be a family of bounded open sets with $M = \bigcup U_i$. Let $\tau_l^{U_i}$ and $\varepsilon_l^{U_i}$ be sequences of numbers from Lemma 8.9 for $\Omega = U_i$.

Let $\{\{\Gamma_t^i\}\}$ be a minimizing sequence of good sweepouts. We will construct a minimizing sequence of good sweepouts $\{\{F_i(\Gamma_t^i)\}\}$ satisfying the conclusions of Lemma 8.8. Maps F_i 's are defined as follows. We set $F_i(\Gamma_t^i)$ to be the hypersurface with $|F_i(\Gamma_t^i)| = \Phi_{U_0} \circ \dots \circ \Phi_{U_i}(|\Gamma_t^i|)$, where Φ_{U_i} is given by Lemma 8.9. (Here we use the standard notation that $|\Sigma|$ denotes the varifold induced by hypersurface Σ).

We claim that for each U_i if there exists a (not relabelled) subsequence $\Gamma_{t_j}^j$ with $\lim_{j \rightarrow \infty} \mathcal{H}^n(F_j(\Gamma_{t_j}^j)) = W_g$ then $\lim_{j \rightarrow \infty} \mathbf{F}_{U_i}(|F_j(\Gamma_{t_j}^j)| \llcorner Gr_n(U_i), \mathcal{V}_{U_i, st}) = 0$. This implies Lemma 8.8.

Fix i . For contradiction suppose there exists a (not relabelled) subsequence $\{F_j(\Gamma_{t_j}^j)\}$ with $\lim_{j \rightarrow \infty} \mathcal{H}^n(F_j(\Gamma_{t_j}^j)) = W_g$ and $\liminf_{j \rightarrow \infty} \mathbf{F}_{U_i}(|F_j(\Gamma_{t_j}^j)| \llcorner Gr_n(U_i), \mathcal{V}_{U_i, st}) > \delta$.

Pick k sufficiently large so that $\frac{1}{2k} + \sum_{l=0}^i \tau_k^l < \delta/2$, where τ_k^l is equal to τ_k from Lemma 8.9 applied to U_l . Let $\bar{\varepsilon} = \frac{1}{2} \min_{l=0, \dots, i} \varepsilon_k^{U_l}$. Fix $j > i$ so that $\mathcal{H}^n(F_j(\Gamma_{t_j}^j)) \in (W_g - \bar{\varepsilon}/10, W_g + \bar{\varepsilon}/10)$.

We have two possibilities. Suppose first that for some $l \in \{0, \dots, i\}$ the varifold $V_l = \Phi_{U_{l+1}} \circ \Phi_{U_{l+2}} \circ \dots \circ \Phi_{U_j}(|\Gamma_{t_j}^j|)$ satisfies $\mathbf{F}_{U_l}(V_l \llcorner Gr_n(U_l), \mathcal{V}_{U_l, st}) > \frac{1}{2k}$. By Lemma 8.9, it follows that $\mathcal{H}^n(F_j(\Gamma_{t_j}^j)) < W_g + \bar{\varepsilon} - \varepsilon_k^{U_l} < W_g - \bar{\varepsilon}$, which contradicts our assumption on $F_j(\Gamma_{t_j}^j)$.

Suppose now that $V_l = \Phi_{U_{l+1}} \circ \Phi_{U_{l+2}} \circ \dots \circ \Phi_{U_j}(|\Gamma_{t_j}^j|)$ satisfies $\mathbf{F}_{U_l}(V_l \llcorner Gr_n(U_l), \mathcal{V}_{U_l, st}) < \frac{1}{2k}$ for all $l \in \{0, \dots, i\}$. We have that $\mathbf{F}_{U_i}(V_i \llcorner Gr_n(U_i), F_j(\Gamma_{t_j}^j) \llcorner Gr_n(U_i)) \leq \sum_{l=0}^i \tau_k^l < \delta/2$ by the triangle equality and the fact that Φ_{U_m} is the identity outside of U_m . From our choice of k we obtain, as a result, that $\mathbf{F}_{U_i}(F_j(|\Gamma_{t_j}^j|) \llcorner Gr_n(U_i), \mathcal{V}_{U_i, st}) < \delta$, giving the desired contradiction.

8.4. Almost minimizing hypersurfaces.

Definition 8.10. (cf. [DT, 3.2]) Given a pair of open sets (U_1, U_2) we call a hypersurface Γ ε -a.m. in (U_1, U_2) if it is ε -a.m. in at least one of the two open sets. Let $\mathcal{CO}(\mathcal{A})$ denote the set of pairs (U_1, U_2) of open sets such that $\inf_{x \in U_1, y \in U_2} d(x, y) \geq 4 \min\{\text{diam}(U_1), \text{diam}(U_2)\}$ and $U_i \in \mathcal{A}$ for $i = 1, 2$.

Recall that $N_r(U) = \{x \in M : d(x, U) < r\}$ denote the r -neighbourhood of U . Let $\mathcal{A}(r, U)$ denote the set of all open subsets V of M , such that either $V \cap cl(U) = \emptyset$ or $V \subset N_r(U)$.

Lemma 8.11. *Let $\{\{\Gamma_t^i\}\}$ be a minimizing sequence of good sweepouts as in Lemma 8.8 and assume furthermore that $\mathcal{H}^n(\Gamma_t^k) < W_g(U) + \frac{1}{8k}$. For every $r > 0$ and N large enough, there exists $t_N \in [0, 1]$ such that*

- $\Gamma^N = \Gamma_{t_N}^N$ is $\frac{1}{N}$ -a.m. in all $(U_1, U_2) \in \mathcal{CO}(\mathcal{A}(r, U))$
- $\mathcal{H}^n(\Gamma^N) \geq W - \frac{1}{N}$
- $\mathcal{H}^n(\Gamma^N \cap cl(N_r(U))) \geq \varepsilon(U)/2$

Proof. The proof is by contradiction (cf. proofs of [CD, 5.3] and [DT, 3.4]). Assume N to be sufficiently large so that $\frac{1}{N} < \varepsilon/2$. Let $A_N = \{t \in [0, 1] : \mathcal{H}^n(\Gamma_t^N) \geq W_g(U) - \frac{1}{N}\}$ and $B_N(U, r) = \{t \in [0, 1] : \mathcal{H}^n(\Gamma_t^N \cap cl(N_r(U))) \geq \varepsilon(U)/2\}$. Define $K_N(U, r) = A_N \cap B_N(U, r)$. $K_N(U, r)$ is a compact set as A_N and $B_N(U, r)$ are closed. By Proposition 2.1 $K_N(U, r)$ is non-empty.

Assume the lemma to be false. Then there is a sequence N_k , so that $\Gamma_t^{N_k}$ is not $\frac{1}{N_k}$ -a.m. in some pair $(U_t^1, U_t^2) \in \mathcal{CO}(\mathcal{A}(r, U))$ for every $t \in K_{N_k}(U, r)$. To simplify notation we will drop sub- and superscript N_k . We will modify family Γ_t on some open set containing $K = K(U, r) \subset [0, 1]$, so that the new family Γ'_t has $\mathcal{H}^n(\Gamma'_t) < W$ for all Γ'_t with $\mathcal{H}^n(\Gamma'_t \cap (U)) > \varepsilon(U)$.

By Lemma 3.1 in [DT] and refinement of the covering argument on page 13 in [DT] it is possible to choose a covering $J_i = (a_i, b_i)$ of K and a collection of sets U_i so that

- each point of K is contained in at most two intervals J_i
- $U_i \in \mathcal{A}(r, U)$ for all i
- if $cl(J_i) \cap cl(J_j) \neq \emptyset$ then $\inf_{x \in U_i, y \in U_j} d(x, y) > 0$
- there exists a $\delta > 0$ such that $\{(a_i + \delta, b_i - \delta)\}$ still cover K and a family $\{\Omega_{i,t}\}$, such that
 - 1) $\Omega_{i,t} = \Omega_t$ if $t \notin J_i$ and $\Omega_{i,t} \setminus U_i = \Omega_t \setminus U_i$ for all t ;
 - 2) $\mathcal{H}^n(\partial \Omega_{i,t}) \leq \mathcal{H}^n(\partial \Omega_t) + \frac{1}{4N}$ for every t ;
 - 3) $\mathcal{H}^n(\partial \Omega_{i,t}) \leq \mathcal{H}^n(\partial \Omega_t) - \frac{1}{2N}$ if $t \in (a_i + \delta, b_i - \delta)$.

We define a new good sweepout $\{\partial \Omega'_t\}$ of U given by

- $\Omega'_t = \Omega_t$ if $t \notin (a_i, b_i)$
- $\Omega'_t = \Omega_{i,t}$ if t is contained in a single J_i
- $\Omega'_t = [\Omega_t \setminus (U_i \cup U_{i+1})] \cup [\Omega_{i,t} \cap U_i] \cup [\Omega_{i+1,t} \cap U_{i+1}]$ if $t \in (a_i, b_i)$

Claim: If $\partial \Omega'_t \cap U \geq \varepsilon$ then $\mathcal{H}^n(\partial \Omega'_t) < W_g(U)$.

By Proposition 2.1 the claim leads to the desired contradiction.

To prove the claim we verify several cases.

Case 1. Suppose $t \notin \bigcup J_i$ then, in particular, $t \notin K$ and $\partial\Omega'_t = \partial\Omega_t$ satisfies $\mathcal{H}^n(\partial\Omega'_t) < W - \frac{1}{N}$ or $\mathcal{H}^n(\partial\Omega'_t \cap U) < \frac{\varepsilon}{2}$.

Case 2. Suppose $t \in \bigcup J_i$, but $t \notin K$. We have two possibilities. Suppose first that $\partial\Omega_t$ satisfies $\mathcal{H}^n(\partial\Omega'_t) < W - \frac{1}{N}$. Since t is contained in at most two distinct intervals J_i we have that $\mathcal{H}^n(\partial\Omega'_t) \leq \mathcal{H}^n(\partial\Omega_t) + 2\frac{1}{4N} < W$. So the claim holds.

Suppose now that $\mathcal{H}^n(\partial\Omega_t \cap N_r(U)) < \varepsilon/2$. We have that t is contained in at most two intervals, say, J_i and J_{i+1} . If U_j (for $j = i$ or $i+1$) intersects both U and its complement then by definition of $\mathcal{A}(r, U)$ we must have $U_j \subset N_r(U)$ and so $\mathcal{H}^n(\partial\Omega_t \cap U_j) < \varepsilon/2$. In other words, mass can be transferred inside U_j from U^{comp} to U , but the transfer can only happen from the part of the hypersurface that lies in $N_r(U)$. It follows that $\mathcal{H}^n(\partial\Omega'_t \cap U) \leq \mathcal{H}^n(\partial\Omega_t \cap N_r(U)) + 2\frac{1}{4N} \leq \varepsilon/2 + 2\frac{1}{4N} < \varepsilon$.

Case 3. Suppose $t \in K$. Since the intervals $\{(a_i + \delta, b_i - \delta)\}$ cover K and each point of K is contained in at most two intervals J_i we have that $\mathcal{H}^n(\partial\Omega'_t) \leq \mathcal{H}^n(\partial\Omega_t) + \frac{1}{4N} - \frac{1}{2N} \leq W - \frac{1}{8N}$

□

Now we can prove Proposition 8.6. Fix $\delta > 0$. Let $\{\Gamma_{t_N}^N\}$ be the min-max sequence from Lemma 8.11. We will show that its subsequence satisfies the requirements of Proposition 8.6. Conditions (B) and (C) are satisfied by construction. We will choose a subsequence that also satisfies (A).

Observe that it follows from the definition if $U \subset V$ and Γ is ε -a.m. in V then Γ is ε -a.m. in U .

Step 1. Almost minimizing annuli around points in $cl(U)$. We start by finding a subsequence of $\{\Gamma_{t_N}^N\}$ that is a.m. for annuli centered at $x \in cl(U)$.

By Lemma 8.11 for each $0 < r < \frac{\delta}{10}$ and each $x \in cl(U)$ we have that Γ^k is $\frac{1}{k}$ -a.m. either in $B_r(x)$ or $N_1(U) \setminus cl(B_{9r}(x))$. For a fixed r as above we have two possibilities.

(a) either $\{\Gamma^k\}$ is $1/k$ -a.m. in $B_r(y)$ for $k > k(y)$ for all $y \in cl(U)$;

(b) or there is a (not relabeled) subsequence $\{\Gamma^k\}$ and a sequence $\{x_r^k\}$, $x_r^k \in cl(U)$, such that Γ^k is $1/k$ -a.m. in $N_1(U) \setminus cl(B_{9r}(x_r^k))$.

Choose a sequence of radii $r_j \rightarrow 0$. If there exists $r_j > 0$ such that (a) holds then condition (A) is satisfied for all $y \in cl(U)$ for $r(y) = \min\{r_j, \delta\}$. Suppose not. By compactness of $cl(U)$ we can select (not relabeled) subsequences $x_{r_j}^k \rightarrow x^j \in cl(U)$ and $x^j \rightarrow x \in cl(U)$. After choosing an appropriate diagonal subsequence we obtain that Γ^k is $\frac{1}{k}$ -a.m. in $N_1(U) \setminus cl(B_{\frac{1}{j}}(x))$ for all $k > j$. In particular, (A) of Proposition 8.5 holds for all annuli centered at x with $r(x) = \delta$. For $y \in cl(U) \setminus x$ we obtain that $\{\Gamma^k\}$ is a.m. for annuli centered at y with $r(y) = \min\{\delta, d(y, x)\}$.

Step 2. Almost minimizing annuli around points in $M \setminus cl(U)$. Let $\{\Gamma^n\}$ denote the min-max sequence from Step 1. By Lemma 8.11 for each $y \in M \setminus cl(U)$ we have that

- (a) either $\{\Gamma^k\}$ is $1/k$ -a.m. in $B_r(y) \setminus cl(U)$ for $k > k(y)$ for all $y \in M \setminus cl(U)$;
- (b) or there is a (not relabeled) subsequence $\{\Gamma^k\}$ and a sequence $\{x_r^k\}$, $x_r^k \in M \setminus cl(U)$, such that Γ^k is $1/k$ -a.m. in $M \setminus cl(U \cup B_{9r}(x_r^k))$.

If (a) holds for some positive radius r_0 then condition (A) is satisfied for all $y \in M \setminus cl(U)$ for $r(y) = \min\{r_0, d(y, cl(U))\}$. Otherwise, we obtain a sequence $\{x^j\}$ and a (not relabeled) subsequence $\{\Gamma^k\}$, such that that Γ^j is $\frac{1}{j}$ -a.m. in $M \setminus cl(U \cup B_{1/j}(x^j))$ for all large j . If sequence $\{x_j\}$ contains a subsequence that converges to a point $x \in M \setminus cl(U)$ then we verify that for a subsequence of $\{\Gamma^k\}$ condition (A) is satisfied for x with $r(x) = d(x, cl(U))$ and for all $y \in M \setminus cl(U)$ with $r(y) = \min\{d(y, cl(U)), d(y, x)\}$. Otherwise there is a subsequence of $\{x_j\}$, such that either $d(x_j, cl(U)) \rightarrow \infty$ or $d(x_j, cl(U)) \rightarrow 0$. In both cases we have that condition (A) is satisfied for all $y \in M \setminus cl(U)$ with $r(y) = d(y, cl(U))$.

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Gregory R. Chambers
Department of Mathematics
University of Chicago
Chicago, Illinois 60637
USA
e-mail: chambers@math.uchicago.edu

Yevgeny Liokumovich
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02142
USA
e-mail: ylio@mit.edu